

Construction of Ricci-type connections by reduction and induction.

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Abstract

Given the Euclidean space \mathbb{R}^{2n+2} endowed with a constant symplectic structure and the standard flat connection, and given a polynomial of degree 2 on that space, Baguis and Cahen [1] have defined a reduction procedure which yields a symplectic manifold endowed with a Ricci-type connection. We observe that any symplectic manifold (M, ω) of dimension $2n$ ($n \geq 2$) endowed with a symplectic connection of Ricci type is locally given by a local version of such a reduction.

We also consider the reverse of this reduction procedure, an induction procedure: we construct globally on a symplectic manifold endowed with a connection of Ricci-type (M, ω, ∇) a circle or a line bundle which embeds in a flat symplectic manifold (P, μ, ∇^1) as the zero set of a function whose third covariant derivative vanishes, in such a way that (M, ω, ∇) is obtained by reduction from (P, μ, ∇^1) .

We further develop the particular case of symmetric symplectic manifolds with Ricci-type connections.

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1 Introduction

A symplectic connection ∇ on a symplectic manifold (M, ω) of dimension $2n$ is a linear connection which is torsion free and for which ω is parallel. The space of symplectic connections on (M, ω) , $\mathcal{E}(M, \omega)$ is infinite dimensional.

Selecting some particular class of connections by curvature conditions has, a priori, two interests. The “moduli space” of such particular connections may be finite dimensional; also, on some compact symplectic manifolds which do admit a connection of the chosen class, this connection may be “rigid”.

In this paper, we describe completely the local behaviour of symplectic connections of Ricci-type (see definition below) and give some global description of simply connected symplectic manifolds admitting a connection of Ricci-type.

We denote by R the curvature of ∇ and by \underline{R} the symplectic curvature tensor

$$\underline{R}(X, Y, Z, T) := \omega(R(X, Y)Z, T).$$

For any point $x \in M$, we have the symmetry properties

- (i) $\underline{R}_x(X, Y, Z, T) = -\underline{R}_x(Y, X, Z, T)$
- (ii) $\underline{R}_x(X, Y, Z, T) = \underline{R}_x(X, Y, T, Z)$
- (iii) $\bigoplus_{X, Y, Z} \underline{R}_x(X, Y, Z, T) = 0.$

From (i) and (ii), $\underline{R}_x \in \Lambda^2 T_x^* M \otimes \odot^2 T_x^* M$, where $\odot^k V$ is the symmetrized k -tensor product of the vector space V .

We denote by r the Ricci tensor of the connection ∇ (i.e. $r_x(X, Y) = \text{tr} [Z \rightarrow R_x(X, Z)Y]$, where X, Y, Z are in $T_x M$); this tensor r is symmetric. We denote by ρ the corresponding endomorphism of the tangent bundle:

$$\omega(X, \rho Y) \stackrel{\text{def}}{=} r(X, Y)$$

so that ρ_x belongs to the symplectic algebra $sp(T_x M, \omega_x)$; in particular $\text{tr } \rho = 0$.

The space \mathcal{R}_x of symplectic curvature tensors at x is

$$\mathcal{R}_x = \ker a \subset \Lambda^2 T_x^* M \otimes \odot^2 T_x^* M$$

where a is the skewsymmetrisation map $a : \Lambda^p T_x^* M \otimes \odot^q T_x^* M \rightarrow \Lambda^{p+1} T_x^* M \otimes \odot^{q-1} T_x^* M$

$$a(u_1 \wedge \dots \wedge u_p \otimes v_1 \dots v_q) := \sum_{i=1}^q u_1 \wedge \dots \wedge u_p \wedge v_i \otimes v_1 \dots \hat{v}_i \dots v_q.$$

The group $Sp(T_x M, \omega_x)$ acts on \mathcal{R}_x . Under this action the space \mathcal{R}_x , in dimension $2n \geq 4$, decomposes into two irreducible subspaces [6]:

$$\mathcal{R}_x = \mathcal{E}_x \oplus \mathcal{W}_x$$

and the decomposition of the curvature tensor \underline{R}_x into its \mathcal{E}_x component (denoted E_x) and its \mathcal{W}_x component (denoted W_x), $\underline{R}_x = E_x + W_x$, is given by

$$E_x(X, Y, Z, T) = -\frac{1}{2(n+1)} [2\omega_x(X, Y)r_x(Z, T) + \omega_x(X, Z)r_x(Y, T) + \omega_x(X, T)r_x(Y, Z) - \omega_x(Y, Z)r_x(X, T) - \omega_x(Y, T)r_x(X, Z)]$$

A connection ∇ is said to be **of Ricci-type** if, at each point x , $W_x = 0$. (Let us mention that such connections were called reducible by Vaisman in [6]). In dimension 2 ($n = 1$), the space \mathcal{W} vanishes identically; so we shall assume in what follows that the manifold has dimension $m = 2n > 2$.

Let us first recall two interesting features of such connections.

- When a symplectic connection is of Ricci-type, it satisfies the equations:

$$\bigoplus_{X, Y, Z} (\nabla_X r)(Y, Z) = 0.$$

Those are the Euler-Lagrange equations of any natural variational principle whose Lagrangian is a second degree invariant polynomial in the curvature (r^2 or R^2). Connections which are solutions of those equations are called preferred ; they are completely described in dimension 2.

- The condition to be of Ricci-type is the condition on a symplectic connection ∇ to have an integrable almost complex structure J^∇ on the twistor space over M which is the bundle of all compatible almost complex structures on M ([2]).

In this paper, we show that any symplectic manifold (M, ω) of dimension $2n$ ($n \geq 2$) admitting a symplectic connection of Ricci type has a local model given by a reduction procedure (as introduced by Baguis and Cahen in [1]) from the Euclidean space \mathbb{R}^{2n+2} endowed with a constant symplectic structure and the standard flat connection.

We also consider the reverse of this reduction procedure, an induction procedure: we construct globally on a simply connected symplectic manifold endowed with a connection of Ricci-type (M, ω, ∇) a circle or a line bundle N which embeds in a flat symplectic manifold (P, μ, ∇^1) as the zero set of a function whose third covariant derivative vanishes, in such a way that (M, ω, ∇) is obtained by reduction from (P, μ, ∇^1) .

We finally describe completely the symmetric symplectic manifolds whose canonical connection is of Ricci-type. Those were already studied in [4] in collaboration with John Rawnsley.

2 Some properties of the curvature of a Ricci-type connection

Let (M, ω) be a smooth symplectic manifold of dim $2n$ ($n \geq 2$) and let ∇ be a smooth Ricci-type symplectic connection. The following results follow directly from the definition (and Bianchi's second identity).

Lemma 2.1 [3] *The curvature endomorphism reads*

$$R(X, Y) = -\frac{1}{2(n+1)}[-2\omega(X, Y)\rho - \rho Y \otimes \underline{X} + \rho X \otimes \underline{Y} - X \otimes \underline{\rho Y} + Y \otimes \underline{\rho X}] \quad (1)$$

where \underline{X} denotes the 1-form $i(X)\omega$ (for X a vector field on M) and where, as before, ρ is the endomorphism associated to the Ricci tensor [$r(U, V) = \omega(U, \rho V)$].

Furthermore:

(i) *there exists a vector field u such that*

$$\nabla_X \rho = -\frac{1}{2n+1}[X \otimes \underline{u} + u \otimes \underline{X}]; \quad (2)$$

(ii) *there exists a function f such that*

$$\nabla_X u = -\frac{2n+1}{2(n+1)}\rho^2 X + fX; \quad (3)$$

(iii) *there exists a real number K such that*

$$\text{tr} \rho^2 + \frac{4(n+1)}{2n+1}f = K. \quad (4)$$

3 Construction by reduction of manifolds with Ricci type connections

Let A be a nonzero element in the symplectic Lie algebra $sp(\mathbb{R}^{2n+2}, \Omega')$ where Ω' is the standard symplectic structure on \mathbb{R}^{2n+2} . Let Σ_A be the closed hypersurface $\Sigma_A \subset \mathbb{R}^{2n+2}$ with equation :

$$\Omega'(x, Ax) = 1; \quad (5)$$

in order for Σ_A to be non empty we replace, if necessary, A , by $-A$.

Let $\dot{\nabla}$ be the standard flat symplectic affine connection on \mathbb{R}^{2n+2} . If X, Y are vector fields tangent to Σ_A define:

$$(\nabla_X^{\Sigma_A} Y)(x) = (\dot{\nabla}_X Y)(x) - \Omega'(AX, Y)x; \quad (6)$$

this is a torsion free linear connection on Σ_A .

The vector field Ax is an affine vector field for this connection; it is clearly complete and we denote by ϕ_t the 1-parametric group of diffeomorphisms of Σ_A generated by this vector field; clearly this flow is given by the restriction to Σ_A of the action of $\exp tA$ on \mathbb{R}^{2n+2} .

Since the vector field Ax is nowhere 0 on Σ_A , for any $x_0 \in \Sigma_A$, there exists :

- a neighborhood $U_{x_0}(\subset \Sigma_A)$,
- a ball $D \subset \mathbb{R}^{2n}$ of radius r_0 , centered at the origin,
- a real interval $I = (-\epsilon, \epsilon)$
- and a diffeomorphism

$$\chi : D \times I \rightarrow U_{x_0} \tag{7}$$

such that $\chi(0,0) = x_0$ and $\chi(y,t) = \phi_t(\chi(y,0))$. We shall denote

$$\pi : U_{x_0} \rightarrow D \quad \pi = p_1 \circ \chi^{-1}.$$

If we view Σ_A as a constraint manifold in \mathbb{R}^{2n+2} , D is a local version of the Marsden-Weinstein reduction of Σ_A around the point x_0 .

If $x \in \Sigma_A$, $T_x \Sigma_A = \langle Ax \rangle^\perp$, where $\langle v_1, \dots, v_p \rangle$ denotes the subspace spanned by v_1, \dots, v_p and $^\perp$ denotes the orthogonal relative to Ω' ; let $\mathcal{H}_x(\subset T_x \Sigma_A) = \langle x, Ax \rangle^\perp$; then

$$T_x \mathbb{R}^{2n+2} = (\mathcal{H}_x \oplus \mathbb{R}Ax) \oplus \mathbb{R}x$$

and π_{*x} defines an isomorphism between \mathcal{H}_x and the tangent space $T_y D$ for $y = \pi(x)$. A vector belonging to \mathcal{H}_x will be called horizontal.

A symplectic form on D , ω , is defined by

$$\omega_y(X, Y) = \Omega'_x(\bar{X}, \bar{Y}) \quad y = \pi(x) \tag{8}$$

where \bar{X} (resp. \bar{Y}) denotes the horizontal lift of X (resp. Y). A symplectic connection ∇ on D is defined by

$$\overline{\nabla_X Y}(x) = \nabla_{\bar{X}}^{\Sigma_A} \bar{Y}(x) + \Omega'(\bar{X}, \bar{Y})Ax \tag{9}$$

Proposition 3.1 [1] *The manifold (D, ω) is a symplectic manifold and ∇ is a symplectic connection of Ricci-type.*

Furthermore, a direct computation shows that the corresponding ρ, u and f are given

by:

$$\overline{\rho X}(x) = -2(n+1)\overline{A_x X} \quad (10)$$

$$\overline{u}(x) = -2(n+1)(2n+1)\overline{A_x^2 x} \quad (11)$$

$$(\pi^* f)(x) = 2(n+1)(2n+1)\Omega'(A^2 x, Ax) \quad (12)$$

where $\overline{A_x^k}$ is the map induced by A^k with values in \mathcal{H}_x :

$$\overline{A_x^k}(X) = A^k X + \Omega'(A^k X, x)Ax - \Omega'(A^k X, Ax)x$$

4 Local models for symplectic connections of Ricci-type

The properties of a symplectic connection of Ricci-type, as stated in Lemma 2.1, imply in particular that

- the curvature tensor is determined by ρ ;
- its covariant derivative is determined by u ;
- its second covariant derivative is determined by ρ and f , hence by ρ and K with K a constant;
- the 3rd covariant derivative of the curvature is determined by u, ρ, K and similarly for all orders.

Hence

Corollary 4.1 *Let (M, ω) be a smooth symplectic manifold of dimension $2n$ ($n \geq 2$) and let ∇ be a smooth Ricci-type connection. Let $p_0 \in M$; then the curvature R_{p_0} and its covariant derivatives $(\nabla^k R)_{p_0}$ (for all k) are determined by (ρ_{x_0}, u_{x_0}, K) .*

Corollary 4.2 *Let (M, ω, ∇) (resp. (M', ω', ∇')) be two real analytic symplectic manifolds of the same dimension $2n$ ($n \geq 2$) each of them endowed with a symplectic connection of Ricci-type.*

Assume that there exists a linear map $b : T_{x_0}M \rightarrow T_{x'_0}M'$ such that (i) $b^\omega'_{x'_0} = \omega_{x_0}$ (ii) $bu_{x_0} = u'_{x'_0}$ (iii) $b \circ \rho_{x_0} \circ b^{-1} = \rho'_{x'_0}$. Assume further that $K = K'$.*

*Then the manifolds are locally affinely symplectically isomorphic, i. e. there exists a normal neighborhood of x_0 (resp. x'_0) U_{x_0} (resp. $U'_{x'_0}$) and a symplectic affine diffeomorphism $\varphi : (U_{x_0}, \omega, \nabla) \rightarrow (U'_{x'_0}, \omega', \nabla')$ such that $\varphi(x_0) = x'_0$ and $\varphi_{*x_0} = b$.*

This follows from classical results, see for instance theorem 7.2 and corollary 7.3 in Kobayashi-Nomizu volume 1 [5].

Consider now (M, ω, ∇) a real analytic symplectic manifold of dimension $2n$ ($n \geq 2$) endowed with an analytic Ricci-type symplectic connection; denote as before by u, ρ, f and K the associated quantities (see lemma 2.1).

Let p_0 be a point in M and choose ξ_0 a symplectic frame of $T_{p_0}M$, i.e. a linear symplectic isomorphism $\xi_0 : (\mathbb{R}^{2n}, \Omega) \rightarrow (T_{p_0}, \omega_{p_0})$, where Ω is the standard symplectic form on \mathbb{R}^{2n} .

Denote by $\tilde{u}(\xi_0)$ the element of \mathbb{R}^{2n} corresponding to $u(p_0)$, i.e.

$$\tilde{u}(\xi_0) = (\xi_0)^{-1} u(p_0)$$

and by $\tilde{\rho}(\xi_0)$ the element of $sp(\mathbb{R}^{2n}, \Omega)$ corresponding to $\rho(p_0)$, i.e.

$$\tilde{\rho}(\xi) = (\xi_0)^{-1} \rho(p_0) \xi_0.$$

Define an element A of $sp(\mathbb{R}^{2n+2}, \Omega')$ as:

$$A = \begin{pmatrix} 0 & \frac{f(p_0)}{2(n+1)(2n+1)} & \frac{-\tilde{u}(\xi_0)}{2(n+1)(2n+1)} \\ 1 & 0 & 0 \\ 0 & \frac{-\tilde{u}(\xi_0)}{2(n+1)(2n+1)} & \frac{-\tilde{\rho}(\xi_0)}{2(n+1)} \end{pmatrix}$$

where $\tilde{u}(\xi_0) = i(\tilde{u}(\xi_0))\Omega$ and where we have chosen a basis $\{e_0, e_{0'}, e_1, \dots, e_{2n}\}$ of the symplectic vector space \mathbb{R}^{2n+2} relative to which the symplectic form has matrix

$$\Omega' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \Omega \end{pmatrix} \quad \Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Consider the local reduction procedure described in section 3 from the element A defined above around the point $x_0 = e_0 \in \Sigma_A = \{x \in \mathbb{R}^{2n+2} \mid \Omega'(x, Ax) = 1\}$.

From what we saw in section 3 this yields a symplectic manifold with a Ricci-type connection (M', ω', ∇') .

Denote by π' the map $\pi' : U_{e_0} \rightarrow M'$ where U_{e_0} is the neighborhood of e_0 in $\Sigma_A \subset \mathbb{R}^{2n+2}$ considered in section 3 and consider $y_0 = \pi'(e_0)$. Then $\mathcal{H}_{e_0} = \langle e_0, Ae_0 = e_{0'} \rangle^\perp = \langle e_1, \dots, e_{2n} \rangle$ is isomorphic under π'_* to $T_{y_0}M'$.

Introduce the injection $j : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n+2}$ with $j(x_1, \dots, x_{2n}) = (0, 0, x_1, \dots, x_{2n})$ so that $j(\mathbb{R}^{2n}) = \mathcal{H}_{e_0}$ and denote by $b : T_{p_0}M \rightarrow T_{y_0}M'$ the map given by

$$b = \pi'_{*e_0} \circ j \circ \xi_0^{-1}.$$

This map b is a linear symplectic isomorphism since

$$\omega'_{y_0}(bX, bY) = \Omega'(j\xi_0^{-1}X, j\xi_0^{-1}Y) = \Omega(\xi_0^{-1}X, \xi_0^{-1}Y) = \omega_{p_0}(X, Y).$$

Furthermore

$$\begin{aligned} u'(y_0) &= \pi'_{*e_0}(\bar{u}'(x_0)) = \pi'_{*e_0}(-2(n+1)(2n+1)(A^2e_0 - \Omega'(A^2e_0, Ae_0)e_0)) \\ &= \pi'_{*e_0}(j\tilde{u}(\xi_0)) = \pi'_{*e_0}(\xi_0)^{-1} u(p_0) = bu(p_0) \\ \rho'(y_0)bX &= \pi'_{*e_0}\overline{\rho'(y_0)X}(e_0) = \pi'_{*e_0}(-2(n+1)\overline{A_{e_0}}(j\xi_0^{-1}(X))) = \pi'_{*e_0}(j\tilde{\rho}(\xi_0)\xi_0^{-1}(X)) \\ &\text{so that } \rho'(y_0)b = b\rho(p_0) \\ (f')(y_0) &= 2(n+1)(2n+1)\Omega'(A^2e_0, Ae_0) = f(p_0). \end{aligned}$$

Hence we have

Theorem 4.3 *Any real analytic symplectic manifold with a Ricci-type connection is locally symplectically affinely isomorphic to the symplectic manifold with a Ricci-type connection obtained by a local reduction procedure around $e_0 = (1, 0, \dots, 0)$ from a constraint surface Σ_A defined by a second order polynomial in the standard flat symplectic manifold $(\mathbb{R}^{2n+2}, \Omega', \dot{\nabla})$.*

5 Construction of a contact manifold which is a global circle or line bundle over M

Consider (M, ω, ∇) a smooth symplectic manifold of dimension $2n > 2$ with a smooth Ricci-type connection and let $B(M) \xrightarrow{\pi} M$ be the $Sp(\mathbb{R}^{2n}, \Omega)$ principal bundle of symplectic frames over M . (An element in the fiber over a point $p \in M$ is a symplectic isomorphism $\xi : (\mathbb{R}^{2n}, \Omega) \rightarrow (T_pM, \omega_p)$).

As before, we consider $\tilde{u} : B(M) \rightarrow \mathbb{R}^{2n}$ the $Sp(\mathbb{R}^{2n}, \Omega)$ equivariant function given by

$$\tilde{u}(\xi) = \xi^{-1}u(x) \text{ where } x = \pi(\xi)$$

and $\tilde{\rho} : B(M) \rightarrow sp(\mathbb{R}^{2n}, \Omega)$ the $Sp(\mathbb{R}^{2n}, \Omega)$ equivariant function given by

$$\tilde{\rho}(\xi) = \xi^{-1}\rho(x)\xi$$

and we define the $Sp(\mathbb{R}^{2n}, \Omega)$ equivariant map $\tilde{A} : B(M) \rightarrow sp(\mathbb{R}^{2n+2}, \Omega')$

$$\tilde{A}(\xi) = \begin{pmatrix} 0 & \frac{(\pi^* f)(\xi)}{2(n+1)(2n+1)} & \frac{-\tilde{u}(\xi)}{2(n+1)(2n+1)} \\ 1 & 0 & 0 \\ 0 & \frac{-\tilde{u}(\xi)}{2(n+1)(2n+1)} & \frac{-\tilde{\rho}(\xi)}{2(n+1)} \end{pmatrix} \quad (13)$$

where $\underline{V} = i(V)\Omega$ for V in \mathbb{R}^{2n} .

We inject the symplectic group $Sp(\mathbb{R}^{2n}, \Omega)$ into $Sp(\mathbb{R}^{2n+2}, \Omega')$ as the set of matrices

$$\tilde{j}(A) = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix} \quad A \in Sp(\mathbb{R}^{2n}, \Omega).$$

Lemma 5.1 *Define the 1-form α on $B(M)$, with values in $sp(\mathbb{R}^{2n+2}, \Omega')$ by:*

$$\alpha_\xi(\overline{X}^{hor}) = \begin{pmatrix} 0 & \frac{-\omega_x(u, X)}{2(n+1)(2n+1)} & \frac{-\widetilde{\rho(X)}(\xi)}{2(n+1)} \\ 0 & 0 & -\tilde{X}(\xi) \\ \tilde{X}(\xi) & \frac{-\widetilde{\rho(X)}(\xi)}{2(n+1)} & 0 \end{pmatrix} \quad (14)$$

where $X \in T_x M$ with $x = \pi(\xi)$ and \overline{X}^{hor} is the horizontal lift of X in $T_\xi B(M)$, and by:

$$\alpha(C^*) = \tilde{j}_*(C) \quad (15)$$

for all $C \in sp(\mathbb{R}^{2n}, \Omega)$ where C^* denotes the fundamental vertical vector field on $B(M)$ associated to C ($C_\xi^* = \frac{d}{dt}\xi \cdot \exp tC|_0$).

This form has the following properties:

- (i) $R_h^* \alpha = \text{Ad}(\tilde{j}(h^{-1})) \alpha \quad \forall h \in Sp(\mathbb{R}^{2n}, \Omega);$
- (ii) $d\tilde{A} = -[\alpha, \tilde{A}];$
- (iii) $d\alpha + [\alpha, \alpha] = -2\tilde{A}\pi^*\omega$

When one has a G -principal bundle $P \xrightarrow{p} M$, an embedding of the group G in a larger group G' , $j : G \rightarrow G'$, and a 1-form α with values in the Lie algebra of G' , such that $\alpha(C^*) = j_*(C)$ for all C in the Lie algebra of G and $R_h^* \alpha = \text{Ad}(j(h^{-1})) \alpha$ for all h in G , one can build the G' -principal bundle $P' = P \times_G G' \xrightarrow{p'} M$ and the unique connection 1-form on P' , α' satisfying $i^* \alpha' = \alpha$ where $i : P \rightarrow P'; \xi \rightarrow [(\xi, 1)]$.

In our situation we build the $Sp(\mathbb{R}^{2n+2}, \Omega')$ - principal bundle

$$B'(M) = B(M) \times_{Sp(\mathbb{R}^{2n}, \Omega)} Sp(\mathbb{R}^{2n+2}, \Omega')$$

whose elements are equivalence classes of pairs (ξ, g) $\xi \in B(M), g \in Sp(\mathbb{R}^{2n+2}, \Omega')$ with (ξ, g) equivalent to $(\xi h, \tilde{j}(h^{-1})g) \quad \forall h \in Sp(\mathbb{R}^{2n}, \Omega)$.

The projection $\pi' : B(M)' \rightarrow M$ maps $[(\xi, g)]$ to $\pi(\xi)$.

The connection 1-form α' is characterised by the fact that

$$\alpha'_{[\xi, 1]}([\overline{X}^{hor}, 0]) = \alpha_\xi(\overline{X}^{hor})$$

and the equations above give:

Lemma 5.2 *The curvature 2-form of the connection 1-form α' is equal to $-2\tilde{A}'\pi'^*\omega$ where \tilde{A}' is the unique $Sp(\mathbb{R}^{2n+2}, \Omega')$ -equivariant extension of \tilde{A} to $B'(M)$.*

This curvature 2-form is invariant by parallel transport ($d^{\alpha'} \text{curv}(\alpha') = 0$).

Thus the holonomy algebra of α' is of dimension 1.

Corollary 5.3 *Assume M is simply connected. The holonomy bundle of α' is a circle or a line bundle over M , $N \xrightarrow{\pi'} M$. This bundle has a natural contact structure ν given by the restriction to $N \subset B(M)'$ of the 1-form $-\alpha'$ (viewed as real valued since it is valued in a 1-dimensional algebra). One has $d\nu = 2\pi'^*\omega$.*

It is enlightening to point out the link between the holonomy bundle N over M and the constraint surface Σ_A when one sees M as obtained (locally) by reduction. The link is only local since Σ_A is in general not a principal bundle over M ; in fact in most cases the quotient of Σ_A by the action of the group $\exp tA$ is at best an orbifold.

Let A be a nonzero element of $sp(\mathbb{R}^{2n+2}, \Omega')$ and let $\Sigma_A = \{y \in \mathbb{R}^{2n+2} \mid \Omega'(y, Ay) = 1\}$; we assume as before that it is not empty. Assume that (M, ω, ∇) is obtained by reduction from Σ_A (as before, we restrict ourselves to some open set in Σ_A).

Let y_0 be a point in Σ_A , let $x_0 = \pi(y_0) \in M$ and choose a symplectic frame ξ_0 at x_0 . Let $\gamma(t)$ be a curve in M such that $\gamma(0) = x_0$. Let $\xi(t)$ be the symplectic frame at $\gamma(t)$ obtained by parallel transport along γ from ξ_0 and let $y(t)$ be the horizontal curve in Σ_A lifting $\gamma(t)$ from y_0 (i.e. $\pi(y(t)) = \gamma(t)$ and $\Omega'(y(t), \dot{y}(t)) = 0$). Define the element $C(t)$ of $Sp(\mathbb{R}^{2n+2}, \Omega')$ as the matrix whose columns are

$$C(t) = \begin{pmatrix} y(t) & Ay(t) & \overline{\xi(t)} \end{pmatrix}$$

where $\overline{\xi(t)}$ consists of the $2n$ vectors which are the horizontal lifts at the point $y(t)$ of the vectors of the frame $\xi(t)$ (the image under the map $\xi(t)$ of the usual basis of \mathbb{R}^{2n}).

Then

$$\frac{d}{dt}C(t)|_s = C(s).\alpha_{\xi(s)}(\overline{\dot{\gamma}(s)}^{hor})$$

where α is the 1-form on $B(M)$ defined in (14) and where \overline{X}^{hor} is the horizontal lift of X in $B(M)$; hence $\overline{\dot{\gamma}(s)}^{hor} = \dot{\xi}(s)$.

Let $B'(M)$ be the $Sp(\mathbb{R}^{2n+2}, \Omega')$ -principal bundle over M considered above and let $[(\xi_0, \Lambda_0)]$ (where Λ_0 is an element in $Sp(\mathbb{R}^{2n+2}, \Omega')$) be a point of $B'(M)$ above x_0 . The horizontal lift of $\gamma(t)$ to $B'(M)$ starting from $[(\xi_0, \Lambda_0)]$ lives in the holonomy subbundle containing this point; it reads

$$[(\xi(t), D(t))]$$

where $\xi(t)$ has been defined above and where $D(t)$ obeys the differential equation

$$\frac{d}{dt}D(t)|_s = -\alpha_{\xi(s)}(\dot{\xi}(s)).D(s)$$

and has initial value Λ_0 .

Define the map above γ which sends $y(t)$ to $[(\xi(t), D(t))]$ where

$$D(t) = C^{-1}(t)C(0)\Lambda_0;$$

this map sends elements of Σ_A to elements in the holonomy bundle through $[(\xi_0, \Lambda_0)]$.

The map from the holonomy bundle through $[(\xi_0, \Lambda_0)]$ to \mathbb{R}^{2n+2} given by:

$$[(\xi, D)] \mapsto C_0\Lambda_0D^{-1}e_0$$

where C_0 is a fixed element in $Sp(\mathbb{R}^{2n+2}, \Omega')$ has value in the hypersurface $\Sigma_{A'}$ where $A' = C_0\tilde{A}(\xi_0)C_0^{-1}$.

6 Embedding of the contact manifold in a flat symplectic manifold

Let (M, ω) be a smooth symplectic manifold of dim $2n$ ($n \geq 2$) and let ∇ be a smooth symplectic connection of Ricci-type. Let (N, α) be a smooth $(2n+1)$ -dimensional contact manifold (i.e. α is a smooth 1-form such that $\alpha \wedge (d\alpha)^n \neq 0$ everywhere). Let X be the corresponding Reeb vector field (i.e. $i(X)d\alpha = 0$ and $\alpha(X) = 1$). Assume there exists a smooth submersion $\pi : N \rightarrow M$ such that $d\alpha = 2\pi^*\omega$. Then at each point $x \in N$, $\text{Ker}(\pi_{*x}) = \mathbb{R}X$ and $\mathcal{L}_X\alpha = 0$.

Remark that such a contact manifold exists always if M is simply connected as we saw in the previous section.

If U is a vector field on M we can define its "horizontal lift" \bar{U} on N by:

$$(i) \quad \pi_* \bar{U} = 0 \quad (ii) \quad \alpha(\bar{U}) = 0.$$

Let us denote by ν the 2-form $\nu = d\alpha = 2\pi^*\omega$ on N . Define a connection ∇^N on N by:

$$\begin{aligned} \nabla_{\bar{U}}^N \bar{V} &= \bar{\nabla}_U \bar{V} - \nu(\bar{U}, \bar{V})X \\ \nabla_X^N \bar{U} &= \nabla_{\bar{U}X}^N \bar{U} = -\frac{1}{2(n+1)}\bar{\rho}\bar{U} \\ \nabla_X^N X &= -\frac{1}{2(n+1)(2n+1)}\bar{u} \end{aligned}$$

where ρ is the Ricci endomorphism of (M, ∇) and where u is the vector field on M appearing in $\nabla\rho$, see lemma 2.1. Then ∇^N is a torsion free connection on N and the Reeb vector field X is an affine vector field for this connection.

The curvature of this connection has the following form:

$$\begin{aligned} R^N(\bar{U}, \bar{V})\bar{W} &= \frac{1}{2(n+1)}[\nu(\bar{\rho}\bar{V}, \bar{W})\bar{U} - \nu(\bar{\rho}\bar{U}, \bar{W})\bar{V}] \\ R^N(\bar{U}, \bar{V})X &= \frac{1}{2(n+1)(2n+1)}[\nu(\bar{u}, \bar{V})\bar{U} - \nu(\bar{u}, \bar{U})\bar{V}] \\ R^N(\bar{U}, X)\bar{V} &= \frac{1}{2(n+1)(2n+1)}\nu(\bar{u}, \bar{V})\bar{U} + \frac{1}{2(n+1)}\nu(\bar{U}, \bar{\rho}\bar{V})X \\ R^N(\bar{U}, X)X &= \frac{1}{2(n+1)(2n+1)}[-\pi^*f\bar{U} + \nu(\bar{U}, \bar{u})X] \end{aligned}$$

where f is the function appearing in lemma 2.1.

Consider now the embedding of the contact manifold N into the symplectic manifold (P, μ) of dimension $2n+2$, where

$$P = N \times \mathbb{R}$$

and, if we denote by s the variable along \mathbb{R} and let $\theta = e^{2s} p_1^* \alpha$ ($p_1 : P \rightarrow N$), we set

$$\mu = d\theta = 2e^{2s} ds \wedge \alpha + e^{2s} d\alpha$$

and let $i : N \rightarrow P$ $x \mapsto (x, 0)$. Obviously $i^*\mu = \nu$.

We now define a connection ∇^1 on P as follows. If Z is a vector field along N , we denote by the same letter the vector field on P such that

$$(i) \quad Z_{i(x)} = i_{*x}Z \quad (ii) \quad [Z, \partial_s] = 0.$$

The formulas for ∇^1 are:

$$\nabla_Z^1 Z' = \nabla^N_Z Z' + \gamma(Z, Z')\partial_s$$

where

$$\begin{aligned}\gamma(Z, Z') &= \gamma(Z', Z) \\ \gamma(X, X) &= \frac{1}{2(n+1)(2n+1)}\pi^* f \\ \gamma(X, \bar{U}) &= -\frac{1}{2(n+1)(2n+1)}\nu(\bar{u}, \bar{U}) \\ \gamma(\bar{U}, \bar{V}) &= \frac{1}{2(n+1)}\nu(\bar{U}, \rho\bar{V})\end{aligned}$$

and

$$\begin{aligned}\nabla_Z^1 \partial_s &= \nabla_{\partial_s}^1 Z = Z \\ \nabla_{\partial_s}^1 \partial_s &= \partial_s.\end{aligned}$$

Theorem 6.1 *The connection ∇^1 on (P, μ) is symplectic and has zero curvature.*

Proposition 6.2 *Let $\psi(s)$ be a smooth function on P . Then ψ has vanishing third covariant differential if and only if*

$$\partial_s^2 \psi - 2\partial_s \psi = 0. \tag{16}$$

In particular the function e^{2s} has this property.

The procedure described above is called the induction.

Let (P, μ, ∇^1) be as above and let $\Sigma = N$ be the constrained submanifold defined by $e^{2s} = 1$. Let Y be the vector field transversal to Σ such that $i(Y)\mu = \alpha$, thus $Y = \partial_s$.

Let H be the 1-parametric group generated by X . Then Σ/H can be identified with M and (M, ω) is the classical Marsden Weinstein reduction of (P, μ) for the constraint Σ .

The connection ∇ on M is obtained from the flat connection ∇^1 on (P, μ) by reduction. Hence

Corollary 6.3 *Any smooth simply connected symplectic manifold with a Ricci-type connection (M, ω, ∇) can be obtained by reduction from an hypersurface Σ in a flat symplectic manifold (P, μ, ∇^1) defined by the 1-level set of a function ψ on P whose third covariant derivative vanishes.*

Corollary 6.4 *Any smooth simply connected symplectic manifold with a Ricci-type connection (M, ω, ∇) is automatically analytic.*

PROOF Since (P, μ, ∇^1) is locally symmetric, P, μ and ∇^1 are real analytic and the explicit construction given preserves analyticity. \square

7 Symmetric symplectic spaces with Ricci-type connections

Lemma 7.1 *The reduction construction described in section 3 yields a locally symmetric symplectic space (i.e. such that the curvature tensor is parallel) if and only if the element $0 \neq A \in sp(\mathbb{R}^{2n+2}, \Omega')$ satisfies $A^2 = \lambda I$ for a constant $\lambda \in \mathbb{R}$.*

PROOF Indeed the connection ∇ has parallel curvature tensor if and only if $\nabla \rho = 0$ hence iff $u = 0$. From the formulas above, this is true iff

$$\overline{A_x^2}(x) = A^2x - \Omega'(A^2x, Ax)x = 0$$

for any $x \in \Sigma_A$. When $u = 0$, f is a constant (cf Lemma 2.1) and it follows from Lemma 3.1 that $\Omega'(A^2x, Ax)$ is a constant λ . Since Σ_A contains a basis of \mathbb{R}^{2n+2} , this yields $A^2 = \lambda I$. \square

Proposition 7.2 *If $0 \neq A \in sp(\mathbb{R}^{2n+2}, \Omega')$ satisfies $A^2 = \lambda I$ for a constant $\lambda \in \mathbb{R}$, the quotient of Σ_A by the action of $\exp tA$ is a manifold and the natural projection map $\Sigma_A \rightarrow M$ is a submersion which endows Σ_A with a structure of circle or line bundle over M .*

PROOF Consider $0 \neq A \in sp(\mathbb{R}^{2n+2}, \Omega')$ so that $A^2 = \lambda I$.

-Case 1: $\lambda > 0$, say $\lambda = k^2$ with $k > 0$.

Then there exists a basis of \mathbb{R}^{2n+2} in which

$$A = \begin{pmatrix} kI_{n+1} & 0 \\ 0 & -kI_{n+1} \end{pmatrix} \quad \Omega' = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

so that $\Sigma_A = \{(u, v) \mid u, v \in \mathbb{R}^{n+1} \mid -2ku \cdot v = 1\}$. The flow of the vector field Ax is given by $\phi_t = e^{tA}$.

The map $\pi : \Sigma_A \rightarrow TS^n = \{(u', v') \mid u', v' \in \mathbb{R}^{n+1} \mid u' \cdot u' = 1, u' \cdot v' = 0\}$ defined by

$$\pi(u, v) = \left(\frac{u}{\|u\|}, \|u\| \left(v + \frac{u}{2k\|u\|^2} \right) \right)$$

induces a diffeomorphism between $M = \Sigma_A/\phi_t$ and TS^n .

M is a non compact simply connected manifold and Σ_A is a \mathbb{R} -bundle over TS^n .

-Case 2 : $\lambda < 0$, say $\lambda = -k^2$ with $k > 0$.

One splits $V^{\mathbb{C}}$ ($V = \mathbb{R}^{2n+2}$) into the eigenspaces relative to A , $V^{\mathbb{C}} = V_{ik} \oplus V_{-ik}$ and observe that those subspaces are Lagrangian. Choosing a basis $\{z_1, \dots, z_{n+1}\}$ for V_{ik} , consider $\omega_{kl} := \Omega'(z_k, \bar{z}_l)$; then $i\omega$ is a Hermitian matrix. A change of basis ($z'_j = \sum z_i U_j^i$) yields $\omega' = {}^t U \omega U$ so we can find a basis for V_{ik} so that $\omega = -2iI_{p,n+1-p} = -2i \begin{pmatrix} I_p & 0 \\ 0 & -I_{n+1-p} \end{pmatrix}$. In the basis of \mathbb{R}^{2n+2} given by $e_j = \frac{1}{2}(z_j + \bar{z}_j)$ $f_j = \frac{1}{2i}(z_j - \bar{z}_j)$ we have:

$$A = \begin{pmatrix} 0 & -kI \\ kI & 0 \end{pmatrix} \quad \Omega' = \begin{pmatrix} 0 & I_{p,n+1-p} \\ -I_{p,n+1-p} & 0 \end{pmatrix}$$

so that $\Sigma_A = \{(u, v) \mid u, v \in \mathbb{R}^{n+1} \mid k \sum_{i \leq p} ((u^i)^2 + (v^i)^2) - k \sum_{i > p} ((u^i)^2 + (v^i)^2) = 1\}$. We assume $p \geq 1$ or replace A by $-A$ so that $\Sigma_A \cong S^{2p-1} \times \mathbb{R}^{2n-2p+2}$ is non empty. The flow ϕ_t is given by the action of $\exp tA = \begin{pmatrix} \cos ktI & -\sin ktI \\ \sin ktI & \cos ktI \end{pmatrix}$.

Then $M = \Sigma_A/\phi_t = (S^{2p-1} \times \mathbb{R}^{2n-2p+2})/U(1)$, so this reduced manifold is:

- $M = \mathbb{R}^{2n}$ if $p = 1$;
- M is a complex line bundle of rank $q := n + 1 - p$ over the complex projective space $P_{p-1}(\mathbb{C}) = S^{2p-1}/U(1)$ if $1 < p \leq n$;
- $M = P_n(\mathbb{C})$ if $p = n + 1$.

In all those cases, M is simply connected and Σ_A is a circle bundle over M ; the only compact case is $M = P_n(\mathbb{C})$.

-Case 3 : $\lambda = 0$, so $A^2 = 0$ with $A \neq 0$. Let us denote by p the rank of A . One splits $V = \mathbb{R}^{2n+2}$ into $V = V_0 \oplus V_1 \oplus V_2$ where $V_1 = \text{Im} A$ ($\dim V_1 = p$), $V_0 \oplus V_1 = \text{Ker} A$ (so $\dim V_0 = 2n + 2 - 2p$ and V_0 is symplectic, since $V_0 \oplus V_1 = V_1^\perp$) and V_2 is a Lagrangian subspace of V_0^\perp supplementary to V_1 . Choose a basis of V_2 and a corresponding basis (dual for Ω') in V_1 and a symplectic basis of V_0 so that in those basis

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A' \\ 0 & 0 & 0 \end{pmatrix} \quad \Omega' = \begin{pmatrix} \Omega_1 & 0 & 0 \\ 0 & 0 & I_p \\ 0 & -I_p & 0 \end{pmatrix}$$

and A' is symmetric. Changing the basis of V_2 and correspondingly the basis of V_1 , one can bring A' to the form $A' = I_{r,p-r}$ so that $\Omega'(x, Ax) = \sum_{i \leq r} (w^i)^2 - \sum_{r < i \leq p} (w^i)^2$ if $x = (u, v, w)$.

Hence $\Sigma_A = S^{r-1} \times \mathbb{R}^{2n+2-r}$ if $r > 1$ and Σ_A consists of two copies of \mathbb{R}^{2n+1} if $r = 1$.

The action of ϕ_t on (u, v, w) is given by $\phi_t(u, v, w) = (u, v + tA'w, w)$ so the reduced manifold is

- two copies of \mathbb{R}^{2n} (if $r = 1$);
- or $M = S^{r-1} \times \mathbb{R}^{2n+1-r}$ if $r > 1$.

In all cases, M is a non compact manifold and Σ_A is a line bundle over M . □

Proposition 7.3 *If $0 \neq A \in sp(\mathbb{R}^{2n+2}, \Omega')$ satisfies $A^2 = \lambda I$ for a constant $\lambda \in \mathbb{R}$, the quotient manifold is a symmetric space and the connection obtained by reduction is the canonical symmetric connection.*

PROOF Any linear symplectic transformation B of \mathbb{R}^{2n+2} which commutes with A obviously induces a symplectic affine transformation $\beta(B)$ of the reduced space $M = \Sigma_A/\phi_t$. If π denotes the canonical projection $\pi : \Sigma_A \rightarrow M$, then

$$\beta(B) \circ \pi = \pi \circ B.$$

In particular the symmetry at the point $x = \pi(y), y \in \Sigma_A$ is induced by

$$B_y u = -u + 2\Omega'(u, Ay)y - 2\Omega'(u, y)Ay.$$

□

We shall now describe the tranvection group of M (i.e. the group G of affine transformations of M generated by the composition of two symmetries). Let us denote by G' the group $G' = \{B \in Sp(\mathbb{R}^{2n+2}, \Omega') \mid BA = AB\}$. The tranvection group of M is clearly included in $\beta(G')$; in fact it is the smallest subgroup of $\beta(G')$ stable under conjugation by a symmetry and which acts transitively on M .

Let $x_0 = \pi(y_0)$ be a point in M and let $s_{x_0} = \beta(B_{y_0})$ be the symmetry at this point. Consider the automorphism of G' given by conjugation by B_{y_0} and denote by σ the induced automorphism of the Lie algebra \mathfrak{g}' of G' . Let $\mathfrak{p}' = \{C \in \mathfrak{g}' \mid \sigma(C) = -C\}$ and $\mathfrak{k}' = \{C \in \mathfrak{g}' \mid \sigma(C) = C\}$.

The dimension of \mathfrak{p}' is equal to $2n$. Indeed, in a basis $\{e_0, e_{0'}, e_1, \dots, e_{2n}\}$ of \mathbb{R}^{2n+2} in which $e_0 = y_0$ and $e_{0'} = Ae_0$ and $\Omega' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \Omega \end{pmatrix}$, one has

$$B_{y_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I_{2n} \end{pmatrix} \quad A = \begin{pmatrix} 0 & \lambda & 0 \\ 1 & 0 & 0 \\ 0 & 0 & A' \end{pmatrix},$$

$$\mathfrak{g}' = \left\{ \begin{pmatrix} b & \lambda c & \underline{A'Z} \\ c & -b & -\underline{Z} \\ Z & A'Z & B \end{pmatrix} \mid b, c \in \mathbb{R}; Z \in \mathbb{R}^{2n}; B \in sp(\mathbb{R}^{2n}, \Omega) \text{ such that } BA' = A'B \right\},$$

$$\text{and } \mathfrak{p}' = \left\{ \begin{pmatrix} 0 & 0 & \underline{A'Z} \\ 0 & 0 & -\underline{Z} \\ Z & A'Z & 0 \end{pmatrix} \mid Z \in \mathbb{R}^{2n} \right\}.$$

Hence the Lie algebra of the transvection group is equal to $\beta_*(\mathfrak{p}' + [\mathfrak{p}', \mathfrak{p}'])$.

In all cases the kernel of β is given by $\exp tA$, and the transvection group is described as follows:

-Case 1: $\lambda > 0$, say $\lambda = k^2$ with $k > 0$.

$$\text{In the basis of } \mathbb{R}^{2n+2} \text{ in which } A = \begin{pmatrix} kI_{n+1} & 0 \\ 0 & -kI_{n+1} \end{pmatrix} \quad \Omega' = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

we have $G' = \left\{ \begin{pmatrix} B & 0 \\ 0 & ({}^tB)^{-1} \end{pmatrix} \mid B \in Gl(n+1, \mathbb{R}) \right\}$, and β of such an element is the identity iff $B = \lambda I$ with $\lambda > 0$.

The transvection group G is isomorphic to $Sl(n+1, \mathbb{R})$ and

$$TS^n = Sl(n+1, \mathbb{R})/Gl(n, \mathbb{R}).$$

-Case 2 : $\lambda < 0$, say $\lambda = -k^2$ with $k > 0$.

$$\text{In the basis of } \mathbb{R}^{2n+2} \text{ in which } A = \begin{pmatrix} 0 & -kI \\ kI & 0 \end{pmatrix} \quad \Omega' = \begin{pmatrix} 0 & I_{p, n+1-p} \\ -I_{p, n+1-p} & 0 \end{pmatrix}$$

we have $G' = \left\{ \begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix} \mid B_1 + iB_2 \in U(p, n+1-p) \right\}$, and β of such an element is the identity iff $B_1 + iB_2 = \exp -ikt$.

The transvection group G is isomorphic to $SU(p, n+1-p)$ and

$$M = SU(p, n+1-p)/U(p-1, n+1-p).$$

-Case 3: $\lambda = 0$, $\text{rank} A = k = p + q$.

$$\text{In the basis of } \mathbb{R}^{2n+2} \text{ in which } A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{pq} \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \Omega' = \begin{pmatrix} \Omega_1 & 0 & 0 \\ 0 & 0 & -I \\ 0 & I & 0 \end{pmatrix}$$

we have $\mathfrak{g}' = \left\{ \begin{pmatrix} D & 0 & C \\ -{}^tC\Omega_1 & -{}^tB & F \\ 0 & 0 & B \end{pmatrix} \mid D \in sp(\mathbb{R}^{2n+2-2k}, \omega_1), B \in so(p, q, \mathbb{R}), F \in gl(k, \mathbb{R}), {}^tF = F, C \in Mat(2n+2-2k, k, \mathbb{R}) \right\}$.

Then \mathfrak{p}' is given by the elements of \mathfrak{g}' for which $D = 0, C = CJ, F = -JFJ, B = -JBJ$ where $J = \begin{pmatrix} 1 & 0 \\ 0 & -I_{k-1} \end{pmatrix}$, so $C = (u \ 0 \ \dots \ 0)$ for $u \in \mathbb{R}^{2n+2-2k}$, $F = \begin{pmatrix} 0 & {}^t v \\ v & 0 \end{pmatrix}$ for $v \in \mathbb{R}^{k-1}$ and $B = \begin{pmatrix} 0 & {}^t w' \\ w & 0 \end{pmatrix}$ for $w \in \mathbb{R}^{k-1}$ and $w' = I_{p-1,q} w$.

Hence $\mathfrak{p}' \oplus [\mathfrak{p}', \mathfrak{p}']$ is the set of all elements in \mathfrak{g}' for which $D = 0$.

The transvection group G has algebra \mathfrak{g} isomorphic to $\{(B, F, C)\}/(0, \mathbb{R}I_{pq}, 0)$ where B is any element in $so(p, q, \mathbb{R})$, F is any symmetric real $k \times k$ matrix, and C is any real $(2n + 2 - 2k) \times k$ matrix and the bracket is defined by

$$[(B, F, C), (B', F', C')] = ([B, B'], -{}^t C \Omega_1 C' + {}^t C' \Omega_1 C - {}^t B F' + {}^t B' F, C B' - C' B),$$

so when $p + q > 2$, the Levi factor is $so(p, q, \mathbb{R})$ and the radical is a 2-step nilpotent algebra. If $p = 0$ and $q = 1$ the transvection group is \mathbb{R}^{2n} and the symmetric space is the standard symplectic vector space. If $p = q = 1$ or if $p = 0$ and $q = 2$, the transvection group is solvable but not nilpotent. The two solvable examples are interesting for building exact quantisation.

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