

Reduction, Induction and Ricci flat symplectic connections.

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We are pleased to dedicate this paper to Hideki Omori on the occasion of his 65th birthday

Abstract

In this paper we present a construction of Ricci-flat connections through an induction procedure. Given a symplectic manifold (M, ω) of dimension $2n$, we define induction as a way to construct a symplectic manifold (P, μ) of dimension $2n + 2$. Given any symplectic connection ∇ on (M, ω) , we define an induced connection ∇^P which is a Ricci-flat symplectic connection on (P, μ) .

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Introduction

A symplectic connection on a symplectic manifold (M, ω) is a torsionless linear connection ∇ on M for which the symplectic 2-form ω is parallel. A symplectic connection exists on any symplectic manifold and the space of such connections is an affine space modelled on the space of symmetric 3-tensorfields on M .

In all what follows, the dimension $2n$ of the manifold M is assumed to be ≥ 4 unless explicitly stated. The curvature tensor R^∇ of a symplectic connection ∇ decomposes [4] under the action of the symplectic group into 2 irreducible components, $R^\nabla = E^\nabla + W^\nabla$. The E^∇ component is defined only in terms of the Ricci-tensor r^∇ of ∇ . All traces of the W^∇ component vanish.

Two particular types of symplectic connections thus arise:

- symplectic connections for which $W^\nabla = 0$; we call them Ricci-type symplectic connections;
- symplectic connections for which $E^\nabla = 0$; they are called Ricci-flat since $E^\nabla = 0 \Leftrightarrow r^\nabla = 0$.

When studying [1] local and global models for Ricci-type symplectic connections, (or more generally [2] so called special symplectic connections), Lorenz Schwachhöfer and the present authors were lead to consider examples of the following construction:

- start with a symplectic manifold (M, ω) of dimension $2n$;
- build a (cooriented) contact manifold (N, α) of dimension $2n + 1$ and a submersion $\pi : N \rightarrow M$ such that $d\alpha = \pi^*\omega$;
- define on the manifold $P = N \times \mathbb{R}$ a natural symplectic structure μ .

It was observed [1] that if (M, ω) admits a symplectic connection of Ricci type one could “lift” this connection to P and the lifted connection is symplectic (relative to μ) and flat.

The aim of this paper is to generalize this result. More precisely we formalize the notion of induction for symplectic manifolds. Starting from a symplectic manifold (M, ω) , we define a contact quadruple (M, N, α, π) , where N, α and π are as above, and we build the corresponding $2n + 2$ dimensional symplectic manifold (P, μ) . We prove the following:

Theorem 4.1 *Let (M, ω) be a symplectic manifold which is the first element of a contact quadruple (M, N, α, π) . Let ∇ be an arbitrary symplectic connection on (M, ω) . Then one can lift ∇ to a symplectic connection on (P, μ) which is Ricci-flat.*

This theorem has various applications. In particular one has

Theorem 5.3 *Let (P, μ) be a symplectic manifold admitting a conformal vector field S which is complete, a symplectic vector field E which commutes with S and assume that, for any $x \in P$, $\mu_x(S, E) > 0$. Assume the reduction of $\Sigma = \{x \in P \mid \mu_x(S, E) = 1\}$ by the flow of E has a manifold structure M with $\pi : \Sigma \rightarrow M$ a surjective submersion. Then (P, μ) admits a Ricci-flat connection.*

The paper is organized as follows. In section 1 we study sufficient conditions for a symplectic manifold (M, ω) to be the first element of a contact quadruple and we give examples of such quadruples. Section 2 is devoted to the lift of hamiltonian (resp conformal) vector fields from (M, ω) to the induced symplectic manifold (P, μ) constructed via a contact quadruple. We show that if (M, ω) is conformal homogeneous, so is (P, μ) . Section 3 describes the structure of conformal homogeneous symplectic manifolds; this part is certainly known but as we had no immediate reference we decided to include it. Section 4 gives some constructions of lifts of symplectic connections of (M, ω) to symplectic connections on the induced symplectic manifold (P, μ) constructed via a contact quadruple. We also prove theorem 4.1. In section 5 we give conditions for a symplectic manifold (P, μ) to be obtained by induction from a contact quadruple (M, N, α, π) . We give also a proof of theorem 5.3.

1 Induction and contact quadruples

Definition 1.1 A **contact quadruple** is a quadruple (M, N, α, π) where M is a $2n$ dimensional smooth manifold, N is a smooth $2n + 1$ dimensional manifold, α is a cooriented contact structure on N (i.e. α is a 1-form on N such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing), $\pi : N \rightarrow M$ is a smooth submersion and $d\alpha = \pi^*\omega$ where ω is a symplectic 2-form on M .

Definition 1.2 Given a contact quadruple (M, N, α, π) the **induced symplectic manifold** is the $2n + 2$ dimensional manifold

$$P := N \times \mathbb{R}$$

endowed with the (exact) symplectic structure

$$\mu := 2e^{2s} ds \wedge p_1^* \alpha + e^{2s} dp_1^* \alpha = d(e^{2s} p_1^* \alpha)$$

where s denotes the variable along \mathbb{R} and $p_1 : P \rightarrow N$ the projection on the first factor.

Induction in the sense of building a $2n + 2$ - dimensional symplectic manifold from a symplectic manifold of dimension $2n$ is also considered by Kostant in [3].

Remark 1.3 • The vector field $S := \partial_s$ on P is such that $i(S)\mu = 2e^{2s}(p_1^*\alpha)$; hence $L_S\mu = 2\mu$ and S is a conformal vector field.

• The Reeb vector field Z on N (i.e. the vector field Z on N such that $i(Z)d\alpha = 0$ and $i(Z)\alpha = 1$) lifts to a vector field E on P such that: $p_{1*}E = Z$ and $ds(E) = 0$. Since $i(E)\mu = -d(e^{2s})$, E is a Hamiltonian vector field on (P, μ) . Furthermore

$$\begin{aligned} [E, S] &= 0 \\ \mu(E, S) &= -2e^{2s}. \end{aligned}$$

- Observe also that if $\Sigma = \{y \in P \mid s(y) = 0\}$, the reduction of (P, μ) relative to the constraint manifold Σ (which is isomorphic to N) is precisely (M, ω) .
- For $y \in P$ define $H_y(\subset T_yP) \Rightarrow E, S \perp^\mu$. Then H_y is symplectic and $(\pi \circ p_1)_{*y}$ defines a linear isomorphism between H_y and $T_{\pi p_1(y)}M$. Vector fields on M thus admit “horizontal” lifts to P .

We shall now make some remarks on the existence of a contact quadruple the first term of which corresponds to a given symplectic manifold (M, ω) .

Lemma 1.4 *Let (M, ω) be a smooth symplectic manifold of dimension $2n$ and let N be a smooth $(2n + 1)$ dimensional manifold admitting a smooth surjective submersion π on M . Let \mathcal{H} be a smooth $2n$ dimensional distribution on N such that $\pi_{*x} : \mathcal{H}_x \rightarrow T_{\pi(x)}M$ is a linear isomorphism (remark that such a distribution may always be constructed by choosing a smooth riemannian metric g on N and setting $\mathcal{H}_x = (\ker \pi_{*x})^\perp$). Then either there exists a smooth nowhere vanishing 1-form α and a smooth vector field Z such that $\forall x \in N$ we have (i) $\ker \alpha_x = \mathcal{H}_x$ (ii) $Z_x \in \ker \pi_{*x}$ (iii) $\alpha_x(Z_x) = 1$ or the same is true for a double cover of N .*

PROOF Choose an auxiliary riemannian metric g on M and consider $N' = \{Z \in TN \mid Z \in \ker \pi_*$ and $g(Z, Z) = 1\}$. If N' has two components, one can choose a global vector field $Z \in \ker \pi_*$ on N and define a smooth 1-form α with $\ker \alpha = \mathcal{H}$ and $\alpha(Z) = 1$. If N' is connected, N' is a double cover of N ($p : N' \rightarrow N : Z_x \rightarrow x$) and we can choose coherently $Z' \in T_ZN'$ by the rule that its projection on T_xN is precisely Z . \square

This says that if we have a pair (M, N) with a surjective submersion $\pi : N \rightarrow M$ we can always assume (by passing eventually to a double cover of N) that there exists a

nowhere vanishing vector field $Z \in \ker \pi_*$ and a nowhere vanishing 1-form α such that $\alpha(Z) = 1$ and $\ker \alpha$ projects isomorphically on the tangent space to M . The vector field Z is determined up to non zero multiplicative factor by the submersion π ; on the other hand, having chosen Z , the 1-form α can be modified by the addition of an arbitrary 1-form β vanishing on Z . If $\tilde{\alpha} = \alpha + \beta$ is another choice, the 2-form $d\tilde{\alpha}$ is the pull back of a 2-form on M iff $i(Z)d\tilde{\alpha} = 0$; i. e. iff:

$$(i) L_Z\beta = -L_Z\alpha \quad (ii) \beta(Z) = 0.$$

This can always be solved locally. We shall assume this can be solved globally.

Lemma 1.5 *Let (M, ω) be a smooth symplectic manifold of dimension $2n$ and let N be a smooth $(2n + 1)$ dimensional manifold admitting a smooth surjective submersion π on M . Let Z be a smooth nowhere vanishing vector field on N belonging to $\ker \pi_*$. Let α be a 1-form such that $\alpha(Z) = 1$. If $L_Z\alpha = \mu\alpha$, for a certain $\mu \in C^\infty(N)$, then $\mu = 0$ and $d\alpha$ is the pullback of a closed 2-form ν on M . Furthermore if X (resp Y) is a vector field on M and \bar{X} (resp \bar{Y}) is the vector field on N such that (i) $\pi_*\bar{X} = X$ (resp. $\pi_*\bar{Y} = Y$) (ii) $\alpha(\bar{X}) = \alpha(\bar{Y}) = 0$ then:*

$$\begin{aligned} [\bar{X}, \bar{Y}] - \overline{[X, Y]} &= -\pi^*(\nu(X, Y))Z \\ [Z, \bar{X}] &= 0. \end{aligned}$$

PROOF We have $\pi_*[Z, \bar{X}] = 0$, $[Z, \bar{X}] = \alpha([Z, \bar{X}])Z = -(L_Z\alpha)(\bar{X})Z = 0$. Since $(L_Z\alpha)(Z) = d\alpha(Z, Z) = 0$, μ vanishes. Also:

$$i(Z)d\alpha = \mathcal{L}_Z\alpha = 0.$$

so $d\alpha$ is the pullback of a closed 2-form ν on M . Finally:

$$\begin{aligned} \pi_*[\bar{X}, \bar{Y}] &= \pi_*\overline{[X, Y]} \\ [\bar{X}, \bar{Y}] &= \overline{[X, Y]} + \alpha([\bar{X}, \bar{Y}])Z = \overline{[X, Y]} - d\alpha(\bar{X}, \bar{Y})Z. \end{aligned}$$

□

Corollary 1.6 *If $\nu = \omega$, the manifold (N, α) is a contact manifold and Z is the corresponding Reeb vector.*

We shall now give examples of contact quadruples for given symplectic manifolds.

Example 1 Let $(M, \omega = d\lambda)$ be an exact symplectic manifold. Define $N = M \times \mathbb{R}$, $\pi = p_1$ (=projection of the first factor), $\alpha = dt + p_1^* \lambda$; then (N, α) is a contact manifold and (M, N, α, π) is a contact quadruple.

The associated induced manifold is $P = N \times \mathbb{R} = M \times \mathbb{R}^2$; with coordinates (t, s) on \mathbb{R}^2 and obvious identification

$$\mu = e^{2s} [d\lambda + 2ds \wedge (dt + \lambda)].$$

Example 2 Let (M, ω) be a quantizable symplectic manifold; this means that there is a complex line bundle $L \xrightarrow{p} M$ with hermitean structure h and a connection ∇ on L preserving h whose curvature is proportional to $i\omega$.

Define $N := \{\xi \in L \mid h(\xi, \xi) = 1\} \subset L$ to be the unit circle sub-bundle. It is a principal $U(1)$ bundle and L is the associated bundle $L = N \times_{U(1)} \mathbb{C}$. The connection 1-form on N (representing ∇) is $u(1) = i\mathbb{R}$ valued and will be denoted α' ; its curvature is $d\alpha' = ik\omega$. Define $\alpha := \frac{1}{ik}\alpha'$ and $\pi := p|_N : N \rightarrow M$ the surjective submersion. Then (M, N, α, π) is a contact quadruple.

The associated induced manifold P is in bijection with $L_0 = L \setminus \text{zero section}$; indeed, consider

$$\Psi : L_0 \rightarrow P = N \times \mathbb{R} : \xi \rightarrow \left(\frac{\xi}{h(\xi, \xi)^{1/2}}, k \ln h(\xi, \xi)^{1/2} \right).$$

Clearly L_0 is a \mathbb{C}^* principal bundle on M ; denote by $\check{\alpha}$ the \mathbb{C}^* -valued 1-form on L_0 representing ∇ ; if $j_1 : N \rightarrow L_0$ is the natural injection and similarly $j_2 : i\mathbb{R} \rightarrow \mathbb{C}$ the obvious injection, we have

$$j_1^* \check{\alpha} = j_2 \circ \alpha'.$$

Then

$$\begin{aligned} ((\Psi^{-1})^* \check{\alpha})_{(\xi_0, s)} (X_{\xi_0} + a\partial_s) &= \check{\alpha}_{\xi_0 e^{s/k}} (\Psi_*^{-1} (X_{\xi_0} + a\partial_s)) \\ &= \check{\alpha}_{\xi_0} (R_{e^{-s/k}*} \circ \Psi_*^{-1} (X_{\xi_0} + a\partial_s)) \\ &= \check{\alpha}_{\xi_0} (X_{\xi_0} + a\partial_s) = j_2 \alpha' (X_{\xi_0}) + \frac{a}{k} \end{aligned}$$

i. e.

$$\Psi^{-1*} \check{\alpha} = p_1^* j_2 \alpha' + \frac{ds}{k}.$$

On the other hand the 1-form $e^{2s} p_1^* \alpha = \frac{1}{ik} e^{2s} p_1^* j_2^* \alpha'$; this shows how the symplectic form $\mu = d(e^{2s} p_1^* \alpha)$ on P is related to the connection form on L_0 [$\mu = d(\frac{e^{2s}}{ik} \Psi^{-1*} \check{\alpha})$].

Such examples have been studied by Kostant [3].

Example 3 Let (M, ω) be a connected homogeneous symplectic manifold; i. e. $M = G/H$ where G is a Lie group which we may assume connected and simply connected and where H is the stabilizer in G of a point $x_0 \in M$. If $p : G \rightarrow M : g \rightarrow gx_0$, $p^*\omega$ is a left invariant closed 2-form on G and $\Omega = (p^*\omega)_e$, (e =neutral element of G) is a Chevalley 2-cocycle on \mathfrak{g} (=Lie Algebra of G) with values in \mathbb{R} (for the trivial representation). Notice that Ω vanishes as soon as one of its arguments is in \mathfrak{h} (=Lie algebra of H). Let $\mathfrak{g}_1 = \mathfrak{g} \oplus \mathbb{R}$ be the central extension of \mathfrak{g} defined by Ω ; i. e.

$$[(X, a), (Y, b)] = ([X, Y], \Omega(X, Y)).$$

Let \mathfrak{h}' be the subalgebra of \mathfrak{g}_1 , isomorphic to \mathfrak{h} , defined by $\mathfrak{h}' := \{(X, 0) \mid X \in \mathfrak{h}\}$. Let G_1 be the connected and simply connected group of algebra \mathfrak{g}_1 , and let H' be the connected subgroup of G_1 with Lie algebra \mathfrak{h}' . **Assume H' is closed.**

Then G_1/H' admits a natural structure of smooth manifold; define $N := G_1/H'$. Let $p_1 : G_1 \rightarrow G$ be the homomorphism whose differential is the projection $\mathfrak{g}_1 \rightarrow \mathfrak{g}$ on the first factor; clearly $p_1(H') \subset H$. Define $\pi : N = G_1/H' \rightarrow M = G/H : g_1H' \mapsto p_1(g_1)H$; it is a surjective submersion.

We shall now construct the contact form α on N : $p_1^* \circ p^*\omega$ is a left invariant closed 2-form on G_1 vanishing on the fibers of $p \circ p_1 : G_1 \rightarrow M$. Its value Ω_1 at the neutral element e_1 of G_1 is a Chevalley 2-cocycle of \mathfrak{g}_1 with values in \mathbb{R} . Define the 1-cochain $\alpha_1 : \mathfrak{g}_1 \rightarrow \mathbb{R} : (X, a) \rightarrow -a$. Then

$$\begin{aligned} \Omega_1((X, a), (Y, b)) &= (p^*\omega)_e(X, Y) = \Omega(X, Y) = -\alpha_1([(X, a), (Y, b)]) \\ &= \delta\alpha_1((X, a), (Y, b)), \end{aligned}$$

i. e. $\Omega_1 = \delta\alpha_1$ is a coboundary. Let $\tilde{\alpha}_1$ be the left invariant 1-form on G_1 corresponding to α_1 . Let $q : G_1 \rightarrow G_1/H' = N$ be the natural projection. We shall show that there exists a 1-form α on N so that $q^*\alpha = \tilde{\alpha}_1$.

For any $U \in \mathfrak{g}_1$ denote by \tilde{U} the corresponding left invariant vector field on G_1 . For any $X \in \mathfrak{h}'$ we have

$$\begin{aligned} i(\tilde{X})\tilde{\alpha}_1 &= \alpha_1(X) = 0 \\ (L_{\tilde{X}}\tilde{\alpha}_1)(\tilde{(Y, b)}) &= -\tilde{\alpha}_1([\tilde{X}, \tilde{(Y, b)}]) = -\alpha_1([X, (Y, b)]) = \Omega(X, Y) = 0 \end{aligned}$$

so that indeed $\tilde{\alpha}_1$ is the pullback by q of a 1-form α on $N = G_1/H'$. Furthermore

$d\alpha = \pi^*\omega$ because both are G_1 invariant 2-forms on N and:

$$\begin{aligned}
(d\alpha)_{q(e_1)}((X, a)^{*N}, (Y, b)^{*N}) &= (q^*d\alpha)_{e_1}(\widetilde{(X, a)}, \widetilde{(Y, b)}) \\
&= (d\tilde{\alpha}_1)_{e_1}(\widetilde{(X, a)}, \widetilde{(Y, b)}) \\
&= \Omega(X, Y), \\
&= \omega_{x_0}(X^{*M}, Y^{*M}) \\
&= (\pi^*\omega)_{q(e_1)}((X, a)^{*N}, (Y, b)^{*N})
\end{aligned}$$

where we denote by U^{*N} the fundamental vector field on N associated to $U \in \mathfrak{g}_1$. Thus

Lemma 1.7 *Let $(M = G/H, \omega)$ be a homogeneous symplectic manifold; let Ω be the value at the neutral element of G of the pull back of ω to G . This is a Chevalley 2 cocycle of the Lie algebra \mathfrak{g} of G . If $\mathfrak{g}_1 = \mathfrak{g} \oplus \mathbb{R}$ is the central extension of \mathfrak{g} defined by this 2 cocycle and G_1 is the corresponding connected and simply connected group let H' be the connected subgroup of G_1 with algebra $\mathfrak{h}' = \{(X, 0) \mid X \in \mathfrak{h}\} \cong \mathfrak{h}$. Assume H' is a closed subgroup of G_1 . Then $N = G_1/H'$ admits a natural submersion π on M and has a contact structure α such that $d\alpha = \pi^*\omega$. Hence $(G/H, G_1/H', \alpha, \pi)$ is a contact quadruple.*

Remark 1.8 The center of G_1 is connected and simply connected, hence the central subgroup $\text{expt}(0, 1)$ is isomorphic to \mathbb{R} . The subgroup $p_1^{-1}(H)$ is a closed Lie subgroup of G_1 whose connected component is $p_1^{-1}(H_0)$ (H_0 =connected component of H). The universal cover $\widetilde{p_1^{-1}(H_0)}$ of $p_1^{-1}(H_0)$ is the direct product of \tilde{H}_0 (=universal cover of H_0) by \mathbb{R} . If $\nu : \widetilde{p_1^{-1}(H_0)} \rightarrow p_1^{-1}(H_0)$ is the covering homomorphism, the subgroup H' we are interested in is $H' = \nu(\tilde{H}_0)$. Clearly if $\pi_1(H_0) \sim \ker \nu$ is finite, H' is closed and the construction proceeds.

2 Lift of hamiltonian vector fields and of conformal vector fields

Let (M, ω) be a symplectic manifold of dimension $2n$ and let (P, μ) be the induced symplectic manifold of dimension $2n + 2$ constructed via the contact quadruple (M, N, α, π) . Let X be a hamiltonian vector field on M ; i. e.

$$\mathcal{L}_X\omega = 0 \quad i(X)\omega = df_X.$$

Consider the horizontal lift \bar{X} of X to N defined by

$$\alpha(\bar{X}) = 0 \quad \pi_*(\bar{X}) = X,$$

and the lift $\bar{\bar{X}}$ of \bar{X} to P defined by

$$p_{1*}\bar{\bar{X}} = \bar{X} \quad ds(\bar{\bar{X}}) = 0.$$

Let Z be the Reeb vector field on (N, α) and let E be its lift to P defined by

$$p_{1*}E = Z \quad ds(E) = 0.$$

Definition 2.1 Define the lift \tilde{X} of a hamiltonian vector field X on (M, ω) as the vector field on P defined by:

$$\tilde{X} = \bar{\bar{X}} - (p_1^*\pi^*f_X) \cdot E =: \bar{\bar{X}} - \widetilde{f_X}E.$$

Lemma 2.2 *The vector field \tilde{X} is a hamiltonian vector field on (P, μ) . Furthermore if \mathfrak{g} is a Lie algebra of vector fields X on M having a strongly hamiltonian action, then the set of vector fields \tilde{X} on P form an algebra isomorphic to \mathfrak{g} and its action on (P, μ) is strongly hamiltonian.*

PROOF $i(\tilde{X})\mu = i(\bar{\bar{X}} - \widetilde{f_X}E)(e^{2s}(p_1^*\pi^*\omega + 2ds \wedge p_1^*\alpha)) = e^{2s}(df_X + 2dsf_X) = d(e^{2s}f_X)$ which shows that \tilde{X} is hamiltonian and that the hamiltonian function is $f_{\tilde{X}} = e^{2s}f_X$.

Also if $X, Y \in \mathfrak{g}$:

$$\begin{aligned} [\tilde{X}, \tilde{Y}] &= [\bar{\bar{X}} - \widetilde{f_X}E, \bar{\bar{Y}} - \widetilde{f_Y}E] \\ &= \overline{[X, Y]} - (\pi \circ p_1)^*\omega(X, Y)E - (\widetilde{Xf_Y})E + \widetilde{Yf_X}E \\ &= \overline{[X, Y]} - (\pi \circ p_1)^*f_{[X, Y]}E \\ &= \widetilde{[X, Y]} \end{aligned}$$

and

$$\{f_{\tilde{X}}, f_{\tilde{Y}}\} = (\bar{\bar{X}} - \widetilde{f_X}E)(e^{2s}\widetilde{f_Y}) = e^{2s}\widetilde{Xf_Y} = e^{2s}f_{\widetilde{[X, Y]}} = f_{\widetilde{[X, Y]}}.$$

□

If C is a conformal vector field on (M, ω) we may assume

$$L_C\omega = \omega \quad di(C)\omega = \omega.$$

By analogy of what we just did, define the lift \tilde{C}_1 of C to (P, μ) by:

$$ds(\tilde{C}_1) = 0 \quad p_{1*}\tilde{C}_1 = \bar{C} + bZ$$

(i.e. $\pi_*p_{1*}\tilde{C}_1 = C$ and $\tilde{C}_1 = \bar{C} + p_1^*bE$). Then

$$\begin{aligned} L_{\tilde{C}_1}\mu &= di(\tilde{C}_1)e^{2s}(p_1^*\pi^*\omega + 2ds \wedge p_1^*\alpha) = d[e^{2s}(p_1^*\pi^*i(C)\omega - 2p_1^*bds)] \\ &= e^{2s}[p_1^*\pi^*\omega + 2ds \wedge p_1^*\pi^*i(C)\omega - 2p_1^*db \wedge ds] \\ &= e^{2s}[p_1^*\pi^*\omega + 2ds \wedge (p_1^*\pi^*(i(C)\omega) + p_1^*db)]. \end{aligned}$$

Thus \tilde{C}_1 is a conformal vector field provided:

$$p_1^*\pi^*i(C)\omega + p_1^*db = p_1^*\alpha.$$

Or equivalently

$$\alpha - \pi^*i(C)\omega = db.$$

The left hand side is a closed 1-form. If this form is exact we are able to lift C to a conformal vector field \tilde{C}_1 on P . Notice that the rate of variation of b along the flow of the Reeb vector field is prescribed:

$$Zb = 1.$$

A variation of this construction reads as follows. Let

$$\tilde{C}_2 = \bar{C} + aE + l\partial_s.$$

Then:

$$\begin{aligned} L_{\tilde{C}_2}\mu &= d(i(\bar{C} + aE + l\partial_s))e^{2s}(p_1^*\pi^*\omega + 2ds \wedge p_1^*\alpha) \\ &= d(e^{2s}(p_1^*\pi^*(i(C)\omega) - 2ads + 2lp_1^*\alpha)) \\ &= e^{2s}[p_1^*\pi^*\omega - 2da \wedge ds + 2lp_1^*\pi^*\omega + 2ds \wedge p_1^*\pi^*i(C)\omega + 2lds \wedge p_1^*\alpha + 2dl \wedge p_1^*\alpha] \\ &= e^{2s}[(1 + 2l)p_1^*\pi^*\omega + 2ds \wedge (da + p_1^*\pi^*i(C)\omega + 2lp_1^*\alpha) + 2dl \wedge p_1^*\alpha]. \end{aligned}$$

If we choose $l = -1/2$

$$L_{\tilde{C}_2}\mu = 2e^{2s}ds \wedge (p_1^*\pi^*i(C)\omega - p_1^*\alpha + da).$$

Thus \tilde{C}_2 is a symplectic vector field on (P, μ) if the closed 1-form $\pi^*i(C)\omega - \alpha$ is exact. If this is the case the lift \tilde{C}_2 is hamiltonian and

$$f_{\tilde{C}_2} = -ae^{2s}.$$

Lemma 2.3 *If C is a conformal vector field on (M, ω) , it admits a lift \tilde{C}_1 (resp. \tilde{C}_2) to (P, μ) which is conformal (resp. hamiltonian) if the closed 1-form $\pi^*i(C)\omega - \alpha$ is exact.*

Let \mathfrak{g} be an algebra of conformal vector fields on (M, ω) . Let $X \in \mathfrak{g}$ be such that $L_{X^*}\omega = \omega$ (where $X_x^* = \frac{d}{dt} \exp -tX.x|_0$; $x \in M$). Then $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_1$, where the vector fields associated to the elements of \mathfrak{g}_1 , are symplectic. We shall assume here that they are hamiltonian; i.e. $\forall Y \in \mathfrak{g}_1, i(Y^*)\omega = df_Y$.

Consider the lifts of these vector fields to (P, μ) .

$$\begin{aligned}
[\tilde{X}_1^*, \tilde{Y}^*] &= [\bar{X}^* + p_1^*bE, \bar{Y}^* - \tilde{f}_Y E] \\
&= [\bar{X}^*, \bar{Y}^*] - p_1^*\pi^*(Xf_Y)E - p_1^*(\bar{Y}^*b)E + \tilde{f}_Y p_1^*(Zb)E \\
&= \overline{[X, Y]}^* + [-p_1^*\pi^*\omega(X, Y) - p_1^*\pi^*\omega(Y, X) + p_1^*\pi^*\omega(X, Y) + \tilde{f}_Y]E \\
&= \overline{[X, Y]}^* + p_1^*\pi^*(\omega(X, Y) + f_Y)E; \\
i([\tilde{X}_1^*, \tilde{Y}^*])\omega &= -L_{Y^*}i(X^*)\omega = -(i(Y^*)d + di(Y^*))i(X^*)\omega \\
&= -i(Y^*)\omega - d\omega(X, Y) = -d(\omega(X, Y) + f_Y).
\end{aligned}$$

Hence

$$[\tilde{X}_1^*, \tilde{Y}^*] = \widetilde{[X^*, Y^*]}.$$

A similar calculation shows that

$$[\tilde{X}_2^*, \tilde{Y}^*] = \widetilde{[X^*, Y^*]}.$$

Notice as before that $L_E\mu = 0$ and $L_{\partial_s}\mu = -2\mu$.

Proposition 2.4 *Let (M, ω) be the first term of a contact quadruple (M, N, α, π) and let (P, μ) be the associated induced symplectic manifold. Then*

- (i) *If G is a connected Lie group acting in a strongly hamiltonian way on (M, ω) , this action lifts to a strongly hamiltonian action of \tilde{G} (= universal cover of G) on (P, μ) .*
- (ii) *If X is a conformal vector field on (M, ω) it admits a conformal (resp. symplectic) lift to (P, μ) if the closed 1-form $\pi^*(i(X)\omega) - \alpha$ is exact. The symplectic lift is in fact hamiltonian.*
- (iii) *The vector field E on P is hamiltonian and the vector field ∂_s is conformal.*

Corollary 2.5 *If (M, ω) admits a transitive hamiltonian action (P, μ) admits a transitive conformal action. If (M, ω) admits a transitive conformal (hamiltonian) action then so does (P, μ) .*

The stability of the class of conformally homogeneous spaces under this construction leads us to the study of these spaces.

3 Conformally homogeneous symplectic manifolds

Definition 3.1 Let (M, ω) be a smooth connected $2n \geq 4$ dimensional symplectic manifold. A connected Lie group G is said to **act conformally on** (M, ω) if

- (i) $\forall g \in G, g^*\omega = c(g)\omega$
- (ii) There exists at least one $g \in G$ such that $c(g) \neq 1$.

As ω is closed $c(g) \in \mathbb{R}$; also $c : G \rightarrow \mathbb{R}$ is a character of G . Let $G_1 = \ker c$; it is a closed, normal, codimension 1 subgroup of G .

Let \mathfrak{g} (resp. \mathfrak{g}_1) be the Lie algebra of G (resp. G_1). Then there exists $0 \neq X \in \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbb{R}X \quad (\text{and}) \quad c_*(X) = 1.$$

The 1-parametric group $\exp tX$ is such that

$$(\exp tX)^*\omega = e^t\omega$$

and this group $\exp tX$ is thus isomorphic to \mathbb{R} . Hence the group G_1 is connected and if G is simply connected so is G_1 . If X^* is the fundamental vector field on M associated to X , remark that $L_{X^*}\omega = -\omega$ since $X_x^* = \frac{d}{dt} \exp -tX \cdot x|_0$.

Definition 3.2 A symplectic manifold (M, ω) of dimension $2n \geq 4$ is called **conformal homogeneous** if there exists a Lie group G acting conformally and transitively on (M, ω) .

We assume M and G connected. Then \tilde{G} ($=$ the universal cover of G) is the semi direct product of \tilde{G}_1 ($=$ the universal cover of G_1) by \mathbb{R} .

By transitivity the orbits of G_1 are of dimension $\geq 2n - 1$. So there are two cases

- (i) The maximum of the dimension of the G_1 orbits is $(2n - 1)$
- (ii) G_1 admits an open orbit.

Case (i) By transitivity the dimension of all G_1 orbits is $(2n - 1)$. If we write as above $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbb{R}X$, the vector field X^* is everywhere transversal to the G_1 orbits. In particular it is everywhere $\neq 0$. Since \mathfrak{g}_1 is an ideal in \mathfrak{g} , the group $\exp tX$ permutes the G_1 orbits.

Clearly if θ_1 is a G_1 orbit, $\bigcup_{t \in \mathbb{R}} \exp tX \cdot \theta_1 = M$.

The restriction $\omega|_{T_x\theta_1}$ has rank $(2n - 1)$. Let Z_x span the radical of $\omega|_{T_x\theta_1}$ and let $\alpha := -i(X^*)\omega \neq 0$ (so $d\alpha = \omega$). As $\alpha_x(Z_x) \neq 0$, we normalize Z_x so that $\alpha_x(Z_x) = 1$. Then

$$T_x M = \mathbb{R}X^* \oplus T_x\theta_1 = \mathbb{R}X^* \oplus (\mathbb{R}Z_x \oplus \ker \underline{\alpha}_x)$$

if $\underline{\alpha}_x = \alpha_x|_{T_x\theta_1}$. If $j : \theta_1 \rightarrow M$ denotes the canonical injection

$$\underline{\alpha} \wedge (d\underline{\alpha})^{n-1} = j^*(\alpha \wedge (\omega)^{n-1}) \neq 0.$$

Thus the orbit θ_1 is a contact manifold, and Z is the Reeb vector field.

Notice that

$$\begin{aligned} (L_{X^*}\alpha)(X^*) &= X^*\alpha(X^*) = 0 \\ (L_{X^*}\alpha)(Y^*) &= X^*\alpha(Y^*) - \alpha([X^*, Y^*]) \\ &= -X^*\omega(X^*, Y^*) + \omega(X^*, [X^*, Y^*]) \\ &= -(L_{X^*}\omega)(X^*, Y^*) = \omega(X^*, Y^*) = -\alpha(Y^*) \end{aligned}$$

for any $Y \in \mathfrak{g}_1$. Hence

$$L_{X^*}\alpha = -\alpha.$$

This says that the various orbits of G_1 have ‘‘conformally’’ equivalent contact structure; i. e.

$$\underline{\alpha}_{\exp tX \cdot x}(\exp tX_* \cdot Y^*) = e^t \underline{\alpha}_x(Y^*) \quad Y \in \mathfrak{G}_1.$$

Furthermore

$$\omega([X^*, Z], Y^*) = X^*\omega(Z, Y^*) - (L_{X^*}\omega)(Z, Y^*) - \omega(Z, [X^*, Y^*]) = 0$$

as $[X^*, Y^*]$ is tangent to the orbit. This says that $[X^*, Z]$ is proportional to Z ; also

$$\alpha([X^*, Z]) = X^*\alpha(Z) - (L_{X^*}\alpha)(Z) = \alpha(Z) = 1$$

hence $[X^*, Z] = Z$ and thus

$$(\exp tX)_* Z_x = e^t Z_{\exp tX \cdot x}$$

Finally

$$\begin{aligned} \alpha([Y^*, Z]) &= -\omega(X^*, [Y^*, Z]) \\ &= -Y^*\omega(X^*, Z) + L_{Y^*}\omega(X^*, Z) + \omega([Y^*, X^*], Z) = 0 \\ \omega([Y^*, Z], Y'^*) &= Y^*\omega(Z, Y'^*) - L_{Y^*}\omega(Z, Y'^*) - \omega(Z, [Y^*, Y'^*]) = 0. \end{aligned}$$

Hence $[Y^*, Z]$ must be proportional to Z and thus

$$[Y^*, Z] = 0$$

which says that the Reeb vector is G_1 stable.

Case (ii) G_1 admits an open orbit. We shall assume that this orbit coincides with M . Thus (M, ω) is a G_1 homogeneous symplectic manifold and ω is exact.

$$\omega = d\eta \quad \text{where} \quad \eta := -i(X^*)\omega.$$

Assume that the action of G_1 is strongly hamiltonian; i. e. $\forall Y \in \mathfrak{g}_1$

$$\begin{aligned} i(Y^*)\omega &= df_Y \\ \{f_Y, f_{Y'}\} &= -\omega(Y^*, Y'^*) = f_{[Y, Y']} \end{aligned}$$

where U^* denotes the fundamental vector field associated to $U \in \mathfrak{g}_1$ on θ_1 . Then

$$\begin{aligned} L_{Y^*}\eta &= -L_{Y^*}i(X^*)\omega = -(L_{Y^*}i(X^*) - i(X^*)L_{Y^*})\omega = -i([Y^*, X^*])\omega \\ &= df_{[X, Y]} = df_{DY} \end{aligned}$$

if $D = \text{ad}X|_{\mathfrak{g}_1}$.

We also have $L_{X^*}\eta = -\eta$.

By Kostant's theorem we may identify M (up to a covering) with a coadjoint orbit θ_1 of G_1 .

Let $\xi \in \theta_1$, let $\pi : G_1 \rightarrow \theta_1 : g_1 \rightarrow g_1 \cdot \xi = \text{Ad}^*g_1\xi$ and let H_1 be the stabilizer of ξ in G_1 . It is no restriction to assume $X_\xi^* = 0$ (since one can replace X by $X + Y$ for any $Y \in \mathfrak{g}_1$ and any tangent vector at ξ can be written in the form Y_ξ^*).

Assuming G (hence G_1) to be connected and simply connected the derivation D exponentiates to a 1-parametric automorphism group of \mathfrak{g}_1 given by e^{tD} and these "exponentiate" to a 1-parametric automorphism group of G_1 which will be denoted $a(t)$. The product law in $G = G_1 \cdot \mathbb{R}$ reads:

$$(g_1, t_1)(g_2, t_2) = (g_1 a(t_1)g_2, t_1 + t_2).$$

As $X_\xi^* = 0$ we have:

$$\begin{aligned} (1, t) \cdot \xi &= \xi \\ (1, t)(g_1, 0) \cdot \xi &= (a(t)g_1, t)\xi = (a(t)g_1, 0)(1, t) \cdot \xi \\ &= (a(t)g_1, 0) \cdot \xi = (a(t)g_1 \circ g_1^{-1}, 0)(g_1, 0) \cdot \xi. \end{aligned}$$

In particular if $g_1 \in H_1$ (= stabilizer of ξ in G_1) $a(t)g_1 \in H_1$; hence if $Y \in \mathfrak{h}_1$ (= Lie algebra of H_1), $[Y, X] \in \mathfrak{h}_1$.

Furthermore

$$\begin{aligned} (L_{X^*}\omega)(Y_1^*, Y_2^*) &= -\omega(Y_1^*, Y_2^*) \\ &= X^*\omega(Y_1^*, Y_2^*) - \omega([X^*, Y_1^*], Y_2^*) - \omega(Y_1^*, [X^*, Y_2^*]). \end{aligned}$$

The above relation at ξ reads:

$$\omega_\xi(Y_1^*, Y_2^*) = \omega_\xi([X, Y_1^*], Y_2^*) + \omega_\xi(Y_1^*, [X, Y_2^*]).$$

But on θ_1 , ω is the Kostant-Souriau symplectic form; hence

$$\begin{aligned} \langle \xi, [Y_1, Y_2] \rangle &= \langle \xi, D[Y_1, Y_2] \rangle \\ \langle \xi - \xi \circ D, [Y_1, Y_2] \rangle &= 0 \end{aligned}$$

That is $\xi - \xi D$ vanishes identically on the derived algebra \mathfrak{g}'_1 .

Conversely suppose we are given an algebra \mathfrak{g}_1 , an element $\xi \in \mathfrak{g}_1^*$ and a derivation D of \mathfrak{g}_1 such that

$$\xi - \xi \circ D \quad \text{vanishes on } \mathfrak{g}'_1.$$

Then, if, as above, H_1 denotes the stabilizer of ξ in G_1 and \mathfrak{h}_1 its Lie algebra, one observes that $Y \in \mathfrak{h}_1$ implies $DY \in \mathfrak{h}_1$.

On the orbit $\theta_1 = G_1 \cdot \xi = G_1/H_1$ define the vector field \hat{X} at $\tilde{\xi} = g_1 \cdot \xi$ by:

$$\hat{X}_{\tilde{\xi}} = \frac{d}{dt} a(-t)g_1 \cdot g_1^{-1} \cdot \tilde{\xi}|_{t=0}.$$

This can be expressed in a nicer way as:

$$\langle \hat{X}_{\tilde{\xi}=g_1\xi}, Z \rangle = \frac{d}{dt} \langle a_{-t}(g_1)g_1^{-1}g_1\xi, Z \rangle|_0 = \frac{d}{dt} \langle a_{-t}(g_1)\xi, Z \rangle|_0$$

for $Z \in \mathfrak{g}_1$

$$\begin{aligned} \text{Ad } a_{-t}(g_1^{-1})Z &= \frac{d}{ds} a_{-t}(g_1^{-1})e^{sZ}a_{-t}(g_1)|_0 = \frac{d}{ds} a_{-t}(g_1^{-1}a_t e^{sZ}g_1)|_0 \\ &= \frac{d}{ds} a_{-t}(g_1^{-1}e^{se^{tD}Z}g_1)|_0 = a_{-t*} \text{Ad } g_1^{-1}e^{tD}Z \\ \frac{d}{dt} \text{Ad } a_{-t}(g_1^{-1})Z|_0 &= -D \circ \text{Ad } g_1^{-1}Z + \text{Ad } g_1^{-1}DZ \end{aligned}$$

i. e.

$$\hat{X}_{\tilde{\xi}=g\xi} = -\xi \circ D \circ \text{Ad } g_1^{-1} + \xi \circ \text{Ad } g_1^{-1} \circ D.$$

Observe that this expression has a meaning; indeed if we assume that $g \in H_1$ (= stabilizer of ξ)

$$\xi \circ \text{Ad } g^{-1} = \xi.$$

Also if $Y \in \mathfrak{h}_1$, $\frac{d}{ds}\langle \xi \circ D \circ \text{Ade}^{-sY} \rangle|_s = -\langle \xi \circ D \circ \text{ad}Y \circ \text{Ade}^{-sY}, Z \rangle = 0$ so that $\langle \xi \circ D \circ \text{Ade}^{-sY}, Z \rangle = \langle \xi \circ D, Z \rangle$.

Thus $\hat{X}_\xi = 0$ and, if $h \in H_1$:

$$\begin{aligned} \hat{X}_{\xi=g \circ \xi=g \cdot h \cdot \xi} &= \xi \circ D \circ \text{Ad } h^{-1} \circ \text{Ad } y^{-1} + \xi \circ \text{Ad } h^{-1} \circ \text{Ad } g^{-1} D \\ &= \xi \circ D \circ \text{Ad } g^{-1} + \xi \circ \text{Ad } g^{-1} D. \end{aligned}$$

Furthermore if $Y \in \mathfrak{g}_1$:

$$[Y^*, \hat{X}]_\xi = (L_{Y^*} \hat{X})_\xi = \frac{d}{dt}(\varphi_{-t^*}^{Y^*} \hat{X}_{\varphi_t^{Y^*} \xi})|_0 = -(DY)_\xi^*.$$

Hence, if $Y_1, Y_2 \in \mathfrak{g}_1$:

$$\begin{aligned} (L_{\hat{X}} \omega)_\xi(Y_1^*, Y_2^*) &= \hat{X}_\xi \omega(Y_1^*, Y_2^*) - \omega((DY_1)^*, Y_2^*) - \omega(Y_1^*, (DY_2)^*) \\ &= \langle -\xi, D[Y_1, Y_2] \rangle = -\langle \xi, [Y_1, Y_2] \rangle = -\omega_\xi(Y_1^*, Y_2^*) \end{aligned}$$

and similarly at any other point, so that \hat{X} is a conformal vector field ($L_{\hat{X}} \omega = -\omega$). We conclude by

Proposition 3.3 *Let (M, ω) be a smooth connected $2n (\geq 4)$ dimensional symplectic manifold which is conformal homogeneous and let G denote the connected component of the conformal group. Then*

- (i) *G admits a codimension 1 closed, connected, invariant subgroup G_1 which acts symplectically on M and $G/G_1 = \mathbb{R}$.*
- (ii) *If the maximum dimension of the G_1 orbits is $(2n - 1)$ M is a union of $(2n - 1)$ dimensional G_1 orbits; each of these orbits is a contact manifold.*
- (iii) *If G_1 acts transitively on M in a strongly hamiltonian way, M is a covering of a G_1 orbit θ in \mathfrak{g}_1^* (= dual of the Lie algebra \mathfrak{g}_1 of G_1). Furthermore if $\xi \in \theta$, there exists a derivation D of \mathfrak{g}_1 such that*

$$\xi - \xi \circ D$$

vanishes on the derived algebra. Conversely if we are given an element $\xi \in \mathfrak{g}_1^$ and a derivation such that $\xi - \xi \circ D$ vanishes on the derived algebra, the orbit θ has the structure of a conformal homogeneous symplectic manifold.*

4 Induced connections

We consider the situation where we have a smooth symplectic manifold (M, ω) of dim $2n$, a contact quadruple (M, N, α, π) and the corresponding induced symplectic manifold (P, μ) .

Let as before Z be the Reeb vector field on the contact manifold (N, α) (i.e. $i(Z)d\alpha = 0$ and $\alpha(Z) = 1$). At each point $x \in N$, $\text{Ker}(\pi_{*x}) = \mathbb{R}Z$ and $L_Z\alpha = 0$.

Recall that $P = N \times \mathbb{R}$ and $\mu = 2e^{2s} ds \wedge p_1^*\alpha + e^{2s} dp_1^*\alpha$ where s is the variable along \mathbb{R} and $p_1 : P \rightarrow N$ the projection on the first factor.

Let ∇ be a smooth symplectic connection on (M, ω) . We shall now define a connection ∇^P on P induced by ∇ .

Let us first recall some notations:

Denote by p the projection $p = \pi \circ p_1 : P \rightarrow M$.

If X is a vector field on M , \bar{X} is the vector field on P such that

$$(i) \quad p_*\bar{X} = X \quad (ii) \quad (p_1^*\alpha)(\bar{X}) = 0 \quad (iii) \quad ds(\bar{X}) = 0.$$

We denote by E the vector field on P such that

$$(i) \quad p_{1*}E = Z \quad (ii) \quad ds(E) = 0.$$

Clearly the values at any point of P of the vector fields $\bar{X}, E, S = \partial_s$ span the tangent space to P at that point and we have

$$[E, \partial_s] = 0 \quad [E, \bar{X}] = 0 \quad [\partial_s, \bar{X}] = 0 \quad [\bar{X}, \bar{Y}] = \overline{[X, Y]} - p^*\omega(X, Y)E.$$

The formulas for ∇^P are:

$$\begin{aligned} \nabla_{\bar{X}}^P \bar{Y} &= \overline{\nabla_X Y} - \frac{1}{2}p^*(\omega(X, Y))E - p^*(\hat{s}(X, Y))\partial_s \\ \nabla_E^P \bar{X} &= \nabla_{\bar{X}}^P E = 2\overline{\sigma X} + p^*(\omega(X, u))\partial_s \\ \nabla_{\partial_s}^P \bar{X} &= \nabla_{\bar{X}}^P \partial_s = \overline{X} \\ \nabla_E^P E &= p^*f \partial_s - 2\overline{U} \\ \nabla_E^P \partial_s &= \nabla_{\partial_s}^P E = E \\ \nabla_{\partial_s}^P \partial_s &= \partial_s \end{aligned}$$

where f is a function on M , U is a vector field on M , \hat{s} is a symmetric 2-tensor on M , and σ is the endomorphism of TM associated to s , hence $\hat{s}(X, Y) = \omega(X, \sigma Y)$.

Notice first that these formulas have the correct linearity properties and yield a torsion free linear connection on P . One checks readily that $\nabla^P \mu = 0$ so that ∇^P is a symplectic connection on (P, μ) .

We now compute the curvature R^{∇^P} of this connection ∇^P . We get

$$R^{\nabla^P}(\bar{X}, \bar{Y})\bar{Z} = \overline{\overline{R^\nabla(X, Y)Z}} + \overline{2\omega(X, Y)\sigma Z - \omega(Y, Z)\sigma X + \omega(X, Z)\sigma Y - \hat{s}(Y, Z)X + \hat{s}(X, Z)Y} + p^*[\omega(X, D(\sigma, U)(Y, Z)) - \omega(Y, D(\sigma, U)(X, Z))]\partial_s$$

$$R^{\nabla^P}(\bar{X}, \bar{Y})E = \overline{2D(\sigma, U)(X, Y) - 2D(\sigma, U)(Y, X)} + p^*[\omega(X, \frac{1}{2}fY - \nabla_Y U - 2\sigma^2 Y) - \omega(Y, \frac{1}{2}fX - \nabla_X U - 2\sigma^2 X)]\partial_s$$

$$R^{\nabla^P}(\bar{X}, E)\bar{Y} = \overline{2D(\sigma, U)(X, Y)} - p^*[\omega(Y, \frac{1}{2}fX - \nabla_X U - 2\sigma^2 X)]\partial_s$$

$$R^{\nabla^P}(\bar{X}, E)E = \overline{2\frac{1}{2}fX - \nabla_X U - 2\sigma^2 X} + p^*[Xf + 4s(X, u)]\partial_s$$

$$\begin{aligned} R^{\nabla^P}(\bar{X}, \bar{Y})\partial_s &= 0 & R^{\nabla^P}(\bar{X}, E)\partial_s &= 0 \\ R^{\nabla^P}(\bar{X}, \partial_s)\bar{Y} &= 0 & R^{\nabla^P}(\bar{X}, \partial_s)E &= 0 & R^{\nabla^P}(\bar{X}, \partial_s)\partial_s &= 0 \\ R^{\nabla^P}(E, \partial_s)\bar{X} &= 0 & R^{\nabla^P}(E, \partial_s)E &= 0 & R^{\nabla^P}(E, \partial_s)\partial_s &= 0 \end{aligned}$$

where

$$D(\sigma, U)(Y, Y') := (\nabla_Y \sigma)Y' + \frac{1}{2}\omega(Y', U)Y - \frac{1}{2}\omega(Y, Y')U.$$

The Ricci tensor r^{∇^P} of the connection ∇^P is given by

$$\begin{aligned} r^{\nabla^P}(\bar{X}, \bar{Y}) &= r^\nabla(X, Y) + 2(n+1)\hat{s}(X, Y) \\ r^{\nabla^P}(\bar{X}, E) &= -(2n+1)\omega(X, u) - 2\text{Tr}[Y \rightarrow (\nabla_Y \sigma)(X)] \\ r^{\nabla^P}(\bar{X}, \partial_s) &= 0 \\ r^{\nabla^P}(E, E) &= 4\text{Tr}(\sigma^2) - 2nf + 2\text{Tr}[X \rightarrow \nabla_X U] \\ r^{\nabla^P}(E, \partial_s) &= 0 \\ r^{\nabla^P}(\partial_s, \partial_s) &= 0 \end{aligned}$$

Theorem 4.1 *In the framework described above, ∇^P is a symplectic connection on (P, μ) for any choice of \hat{s}, U and f . The vector field E on P is affine ($L_{\bar{E}}\nabla^P = 0$) and symplectic ($L_{\bar{E}}\mu = 0$); the vector field ∂_s on P is affine and conformal ($L_{\partial_s}\mu = 2\mu$).*

Furthermore, choosing

$$\hat{s} = \frac{-1}{2(n+1)}r^\nabla$$

$$\begin{aligned}\underline{U} &:= \omega(U, \cdot) = \frac{2}{2n+1} \text{Tr}[Y \rightarrow \nabla_Y \sigma] \\ f &= \frac{1}{2n(n+1)^2} \text{Tr}(\rho^\nabla)^2 + \frac{1}{n} \text{Tr}[X \rightarrow \nabla_X U].\end{aligned}$$

we have:

- the connection ∇^P on (P, μ) is Ricci flat (i.e. has zero Ricci tensor);
- if the symplectic connection ∇ on (M, ω) is of Ricci type, then the connection ∇^P on (P, μ) is flat.
- if the connection ∇^P is locally symmetric, the connection ∇ is of Ricci type, hence ∇^P is flat.

PROOF The first point is an immediate consequences of the formulas above for r^{∇^P} . The second point is a consequence of the differential identities satisfied by the Ricci type symplectic connections (which appear in M. Cahen, S. Gutt, J. Horowitz and J. Rawnsley, Homogeneous symplectic manifolds with Ricci-type curvature, *J. Geom. Phys.* **38** (2001) 140–151).

The third point comes from the fact that $(\nabla_{\frac{P}{2}} R^{\nabla^P})(\bar{X}, \bar{Y})\bar{T}$ contains only one term in E whose coefficient is $\frac{1}{2}W^{\nabla^P}(X, Y, T, Z)$. \square

5 A reduction construction

We present here a procedure to construct symplectic connections on some reduced symplectic manifolds; this is a generalisation of the construction given by P. Baguis and M. Cahen [Lett. Math. Phys. 57 (2001), pp. 149-160].

Let (P, μ) be a symplectic manifold of dimension $(2n+2)$. Assume P admits a complete conformal vector field S :

$$L_S \mu = 2\mu; \quad \text{define } \alpha := \frac{1}{2}i(S)\mu \quad \text{so that } d\alpha = \mu.$$

Assume also that P admits a symplectic vector field E commuting with S

$$L_E \mu = 0 \quad [S, E] = 0 \quad (\Rightarrow L_E \alpha = 0).$$

Then $S\mu(S, E) = (L_S \mu)(S, E) = 2\mu(S, E)$, so if x is a point of P where $\mu_x(S, E) \neq 0$ and if s is a parameter along the integral line γ of S passing through x and taking value

0 at x , we have $\mu_{\gamma(s)}(S, E) = e^{2s}\mu_x(S, E)$.

Assume $P' := \{x \in P \mid \mu_x(S, E) > 0\} \neq \emptyset$ and let:

$$\Sigma = \{x \in P \mid \mu_x(S, E) = 1\} = \{x \in P \mid f_E(x) = \frac{1}{2}\}$$

where $f_E = -i(E)\alpha = -\frac{1}{2}\mu(S, E)$ so that $df_E = -L_E\alpha + i(E)d\alpha = i(E)\mu$.

Thus $\Sigma \neq \emptyset$ and it is a closed hypersurface (called the constraint hypersurface). Remark that $P' \cong \Sigma \times \mathbb{R}$.

The tangent space to the hypersurface Σ is given by

$$T_x\Sigma = \ker(df_E)_x = \ker(i(E)\mu)_x = E^\perp.$$

The restriction of μ_x to $T_x\Sigma$ has rank $2n - 2$ and a radical spanned by E_x .

Remark thus that the restriction of α to Σ is a contact 1-form on Σ .

Let \sim be the equivalence relation defined on Σ by the flow of E . Assume that the quotient Σ/\sim has a $2n$ dimensional manifold M structure so that $\pi : \Sigma \rightarrow \Sigma/\sim = M$ is a smooth submersion.

Define on Σ a ‘‘horizontal’’ distribution of dimension $2n$, \mathcal{H} , by

$$\mathcal{H} = \langle E, S \rangle^{\perp\mu},$$

and remark that $\pi_{*|\mathcal{H}_y} : \mathcal{H}_y \rightarrow T_{x=\pi(y)}M$ is an isomorphism.

Define as usual the reduced 2-form ω on M by

$$\omega_{x=\pi(y)}(Y_1, Y_2) = \mu_y(\bar{Y}_1, \bar{Y}_2)$$

where \bar{Y}_i ($i = 1, 2$) is defined by (i) $\pi_*\bar{Y}_i = Y_i$ (ii) $\bar{Y}_i \in \mathcal{H}_y$.

Notice that $\pi_*[E, \bar{Y}] = 0$, and $\mu(S, [E, \bar{Y}]) = -L_E\mu(S, \bar{Y}) + E\mu(S, \bar{Y}) = 0$ hence

$$[E, \bar{Y}] = 0.$$

The definition of ω_x does not depend on the choice of y . Indeed

$$E\mu(\bar{Y}_1, \bar{Y}_2) = L_E\mu(\bar{Y}_1, \bar{Y}_2) + \mu([E, \bar{Y}_1], \bar{Y}_2) + \mu(\bar{Y}_1, [E, \bar{Y}_2]) = 0.$$

Clearly ω is of maximal rank $2n$ as \mathcal{H} is a symplectic subspace. Finally

$$\begin{aligned} \pi^*(d\omega(Y_1, Y_2, Y_3)) &= \bigoplus_{123} (Y_1\omega(Y_2, Y_3) - \omega([Y_1, Y_2], Y_3)) \\ &= \bigoplus_{123} (\bar{Y}_1\mu(\bar{Y}_2, \bar{Y}_3) - \mu(\overline{[Y_1, Y_2]}, \bar{Y}_3)) \end{aligned}$$

and

$$[\bar{Y}_1, \bar{Y}_2] = \overline{[Y_1, Y_2]} + \mu(S, [\bar{Y}_1, \bar{Y}_2])E.$$

Hence ω is closed and thus symplectic. Clearly $\pi^*\omega = \mu|_\Sigma = d(\alpha|_\Sigma)$.

Remark 5.1 The symplectic manifold (M, ω) is the first element of a contact quadruple $(M, \Sigma, \frac{1}{2}\alpha|_{\Sigma}, \pi)$ and the associated symplectic $(2n+2)$ -dimensional manifold is $(P', \mu|_{P'})$.

We shall now consider the reduction of a connection. Let $(P, \mu), E, S, \Sigma, M, \omega$ be as above. Let ∇^P be a symplectic connection on P and assume that the vector field E is affine ($L_E \nabla^P = 0$).

Then define a connection ∇^Σ on Σ by

$$\nabla_A^\Sigma B := \nabla_A^P B - \mu(\nabla_A^P B, E)S = \nabla_A^P B + \mu(B, \nabla_A^P E)S.$$

Then:

$$\begin{aligned} \nabla_A^\Sigma B - \nabla_B^\Sigma A - [A, B] &= (\mu(B, \nabla_A^P E) - \mu(A, \nabla_B^P E))S \\ &= (\mu(B, \nabla_E^P A + [A, E]) - \mu(A, \nabla_E^P B + [B, E]))S \\ &= (E\mu(B, A) - \mu(B, [E, A]) - \mu([E, B], A))S \\ &= (L_E \mu(B, A))S = 0. \end{aligned}$$

Also

$$\begin{aligned} (L_E \nabla^\Sigma)_A B &= [E, \nabla_A^P B + \mu(B, \nabla_A^P E)S] \\ &\quad - \nabla_{[E, A]}^P B - \mu(B, \nabla_{[E, A]}^P E)S - \nabla_A^P [E, B] - \mu([E, B], \nabla_A^P E)S \\ &= (L_E \nabla^P)_A B + (E\mu(B, \nabla_A^P E) - \mu(B, \nabla_{[E, A]}^P E) - \mu([E, B], \nabla_A^P E))S \\ &= (L_E \mu)(B, \nabla_A^P E) + \mu(B, [E, \nabla_A^P E] - \nabla_{[E, A]}^P E)S = 0 \end{aligned}$$

i. e. ∇^Σ is a torsion free connection and E is an affine vector field for ∇^Σ .

Define a connection ∇^M on M by:

$$\overline{\nabla_{\bar{Y}_1}^M Y_2}(y) = \nabla_{\bar{Y}_1}^\Sigma \bar{Y}_2(y) - \mu(\bar{Y}_2, \nabla_{\bar{Y}_1}^P S)E.$$

If $x \in M$, this definition does not depend on the choice of $y \in \pi^{-1}(x)$. Also

$$\begin{aligned} \overline{\nabla_{\bar{Y}_1}^M Y_2} - \overline{\nabla_{\bar{Y}_2}^M Y_1} - \overline{[Y_1, Y_2]} &= \nabla_{\bar{Y}_1}^\Sigma \bar{Y}_2 - \nabla_{\bar{Y}_2}^\Sigma \bar{Y}_1 - \overline{[Y_1, Y_2]} \\ &\quad + (-\mu(\bar{Y}_2, \nabla_{\bar{Y}_1}^P S) + \mu(\bar{Y}_1, \nabla_{\bar{Y}_2}^P S))E \\ &= \mu(S, [\bar{Y}_1, \bar{Y}_2])E + (\mu(\nabla_{\bar{Y}_1}^P \bar{Y}_2, S) - \mu(\nabla_{\bar{Y}_2}^P \bar{Y}_1, S))E = 0 \end{aligned}$$

Finally

$$\begin{aligned} \pi^*((\nabla_{\bar{Y}_1}^M \omega)(Y_2, Y_3)) &= \pi^*(Y_1 \omega(Y_2, Y_3) - \omega(\nabla_{\bar{Y}_1}^M Y_2, Y_3) - \omega(Y_2, \nabla_{\bar{Y}_1}^M Y_3)) \\ &= \bar{Y}_1 \mu(\bar{Y}_2, \bar{Y}_3) - \mu(\nabla_{\bar{Y}_1}^P \bar{Y}_2 + \mu(\bar{Y}_2, \nabla_{\bar{Y}_1}^P E)S - \mu(\bar{Y}_2, \nabla_{\bar{Y}_1}^P S)E, \bar{Y}_3) \\ &\quad - \mu(\bar{Y}_2, \nabla_{\bar{Y}_1}^P \bar{Y}_3 + \mu(\bar{Y}_3, \nabla_{\bar{Y}_1}^P E)S - \mu(\bar{Y}_3, \nabla_{\bar{Y}_1}^P S)E) \\ &= 0 \end{aligned}$$

i. e. the connection ∇^M is symplectic.

Lemma 5.2 *Let (P, μ) be a symplectic manifold admitting a symplectic connection ∇^P , a conformal vector field S which is complete, a symplectic vector field E which is affine and commutes with S . If the constraint manifold $\Sigma = \{x \in P \mid \mu_x(S, E) = 1\}$ is not empty, and if the reduction of Σ is a manifold M , this manifold admits a symplectic structure ω and a natural reduced symplectic connection ∇^M .*

In particular

Theorem 5.3 *Let (P, μ) be a symplectic manifold admitting a conformal vector field S ($L_S \mu = 2\mu$) which is complete, a symplectic vector field E which commutes with S and assume that, for any $x \in P$, $\mu_x(S, E) > 0$. If the reduction of $\Sigma = \{x \in P \mid \mu_x(S, E) = 1\}$ by the flow of E has a manifold structure M with $\pi : \Sigma \rightarrow M$ a surjective submersion, then M admits a reduced symplectic structure ω and (P, μ) is obtained by induction from (M, ω) using the contact quadruple $(M, \Sigma, \frac{1}{2}i(S)\mu|_{\Sigma}, \pi)$.*

In particular (P, μ) admits a Ricci-flat connection.

Reducing (P, μ) as above and inducing back we see that theorem 4.1 immediately proves this.

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