Deformation Quantization and Symmetries
Episode 5

Simone Gutt, ULB

Programme

Introduction to the concept of deformation quantization (existence, classification and representation results for formal star products).

Notion of formal star products with symmetries; one has a Lie group action (or a Lie algebra action) compatible with the classical Poisson structure, and one wants to consider star products such that the Lie group acts by automorphisms (or the Lie algebra acts by derivations). We recall in particular the link between left invariant star products on Lie groups and Drinfeld twists, and the notion of universal deformation formulas.

Quantum moment map : Classically, symmetries are particularly interesting when they are implemented by a moment map and we give indications to build a corresponding quantum version.

Quantum reduction : reduction is a construction in classical mechanics with symmetries which allows to reduce the dimension of the manifold; we describe one of the various quantum analogues which have been considered in the framework of formal deformation quantization.

Considerations about convergence of star products .

The need for more

For physics, $\hbar$ is a constant of nature and $\nu = i\hbar$ is not a formal parameter. Formal deformation is not enough; for instance, there is no general reasonable notion of spectra for formal star product algebras. Spectra can be recovered only for a few examples with convergence.. In general, formal deformation quantization can not predict the values one would obtain by measurements.

In non formal deformation quantization of a Poisson manifold, one would like to have a subalgebra $\mathcal{A}$ of complex valued smooth functions (or distributions) on the manifold, with some topology, and a family of continuous associative law $*_\lambda$ on $\mathcal{A}$, depending on a parameter $\hbar$ belonging to a set I admitting 0 in its closure, so that the limit of $*_\hbar$ when $\hbar \mapsto 0$ is the usual product, and the limit of the $\frac{[\cdot,\cdot]_{*_\hbar}}{\hbar}$ is the Poisson bracket. One would also like the topology to be such that one could define nice representations of $(\mathcal{A}, *_\lambda)$ and spectra.

It is well known that the framework of C^*-algebras provides a nice background for a notion of spectra (the spectrum of an element a in a unital C^*-algebra is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda 1$ is not invertible), but this framework might be too restrictive.

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Weyl quantization

The Moyal star product presents interesting features concerning convergence. Recall that the formal Moyal star product comes from the quantization of polynomials on $\mathbb{R}^{2n}$ with Weyl's ordering. Weyl quantization can be extended beyond polynomials; heuristically one would like

$$\text{“}\mathcal{Q}_{Weyl}(F)\text{”} = (\frac{1}{2\pi})^{2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{F}(u,v) e^{i(uQ+vP)} dudv$$

where $\hat{F}$ is the Fourier transform $\hat{F}(u,v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(q,p) e^{-i(uq+vp)} dqdp$. If one develops formally this, using the fact that on a nice test function ϕ, $(e^{iuQ}\phi)(x) = e^{iu.x}\phi(x)$, $(e^{ivP}\phi)(x) = \phi(x+\hbar v)$ and $e^{i(uQ+vP)} = e^{\frac{i}{2}\hbar u.v} e^{iuQ} \circ e^{ivP}$, one gets the formula $(\text{“}\mathcal{Q}_{Weyl}(F)\text{”}(\phi))(x) = \frac{1}{(2\pi)^{2n}} \int\int\int\int_{(\mathbb{R}^n)^{\otimes 4}} F(q,p) e^{-isq-itp} e^{i\hbar st/2} e^{isx} \phi(x+\hbar t) dsdtdqdp$.

If $t = \frac{y-x}{\hbar}$, we get $\frac{1}{(2\pi)^{2n}\hbar^n} \int\int\int\int_{(\mathbb{R}^n)^{\otimes 4}} F(q,p) e^{-isq-i(y-x)p/\hbar} e^{is(x+y)/2} \phi(y) dsdydqdp$, which is $\frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(\frac{x+y}{2}, p) e^{-i(y-x)p/\hbar} \phi(y)\, dydp$. Setting $p = 2\pi\hbar\xi$, it gives

$$(\mathcal{Q}_{Weyl}(F)(\phi))(x) := \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F(\frac{x+y}{2}, 2\pi\hbar\xi) e^{-2\pi i(y-x)\xi} \phi(y) dy \right) d\xi;$$

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Moyal convergent star product

• The map $\mathcal{Q}_{Weyl}$ gives an isometry between the space $L^2(\mathbb{R}^{2n})$ and the space of Hilbert Schmidt operators on $L^2(\mathbb{R}^n)$, associating a self-adjoint operator to a real function.

• If F and G are two Schwartz functions, then the composition of the corresponding operators $\mathcal{Q}_{Weyl}(F) \circ \mathcal{Q}_{Weyl}(G)$ is equal to $\mathcal{Q}_{Weyl}(F *_\hbar G)$ where $F *_\hbar G$ is the function defined by

$$(F *_\hbar G)(u) := (\frac{1}{\pi\hbar})^{2n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{2i}{\hbar}\Omega(v,w)} F(u+v)G(u+w)dvdw \quad (1)$$

$$= (\frac{1}{\pi\hbar})^{2n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{2i}{\hbar}(\Omega(u,v)+\Omega(v,w)+\Omega(w,u))} F(v)G(w)dvdw. \quad (2)$$

with $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The result is a Schwartz function;hence $*_\hbar$ gives an associative product on the space of Schwartz functions, called the **convergent Moyal star product**.

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Integral formulas for star products

Many examples of star products are related to integral formulas. For instance, the Berezin or Toeplitz star product on Kähler manifolds are obtained as asymptotic expansions for $\hbar \to 0$ of some convergent counterpart in usual quantization, given by an integral formula.

For instance, if (M, ω, J) is a compact Kähler manifold and (L, ∇, h) is a regular quantization bundle over M, the Berezin's symbols were defined previously
$\hat{A}(x) := \frac{<\mathbf{A}e_q, e_q>}{\|e_q\|^2} \quad q \in L_0, \pi(q) = x \in M$ with coherent states defined by
$s(\pi(q)) =< s, e_q > q \quad \text{for any} \quad s \in \mathcal{H}$; and the formula for the composition of Berezin's symbols is given by

$$(A *_k B)(x) = \int_M \hat{A}(x,y)\hat{B}(y,x)\psi^k(x,y)\epsilon^{(k)}\frac{k^n\omega^n}{n!} \qquad A, B \in \hat{E}(L^l), k \geq l \tag{3}$$

where $\psi(x,y) = \frac{|\langle e_{q'}, e_q\rangle|^2}{\|e_{q'}\|^2\|e_q\|^2}$ with $\pi(q) = x$ and $\pi(q') = y$.

The asymptotic expansion in k^{-1} as $k \to \infty$ is well defined; it gives a series in $\frac{1}{k}$ which is a differential star product on the manifold.

The difficulty to get convergent deformations in this framework of an integral formula depending on a parameter k (given an associative law $*_k$ on a space $\hat{E}(L^k)$) is to find an algebra, i.e. a subspace stable by all $*_k$.

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The disk

Berezin's procedure can be extended to non compact Kähler manifolds. For a possibly unbounded operator **A** to have a Berezin's symbol, the coherent states must be in the domain, $\mathbf{A}e_q \in \mathcal{H}$, $\forall q$; do be able to write a composition formula $\mathbf{A} \circ \mathbf{B}$ in terms of symbols as above, one needs the adjoint of **A** to be defined on coherent states
(so the section $s(x) =< e_{q'}, \mathbf{A}e_q > q$ should be holomorphic and square integrable for all q') and one needs all $\mathbf{B}e_q$ to be in the domain of **A**.

Consider the open disk, $\left(\mathbb{D}, \omega = \frac{-i\lambda dz\wedge d\bar{z}}{2\pi(1-|z|^2)^2} = d\left(\frac{i\lambda\bar{z}dz}{2\pi(1-|z|^2)}\right)\right)$; then $\mathbb{D} = SU(1,1)/U(1)$ and the action of $SU(1,1)$ is Hamiltonian.

If (L, ∇, h) is a homogenous quantization for the simply-connected group $\widetilde{SU(1,1)}$ then L can be trivialised on all of $\mathbb{D}$ by a section s_0 with $|s_0|^2 = (1-|z|^2)^\lambda$. The norm on holomorphic sections is

$$||fs_0||^2 = \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\lambda \frac{\lambda d^2 z}{\pi(1-|z|^2)^2}$$

where d^2z denotes the usual Lebesgue measure; $||s_0||^2$ is finite for $\lambda > 1$ which we assume.
The characteristic function is $\epsilon = 1 - \lambda^{-1}$

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A class of symbols on the disk

The class of symbols which we shall use are the symbols of differential operators $D(p,q,k)$ on L^k defined by

$$D(p,q,k)(fs_0^k)(z) = \left\{ z^p \left(\frac{\partial}{\partial z} \right)^q f(z) \right\} s_0^k(z).$$

We have $D(p,q,k)e^{(k)}_{s_0(w)}(z) = \epsilon^{(k)} P_q(k\lambda) z^p \left(\frac{\overline{w}}{1-\overline{w}z} \right)^q (1-\overline{w}z)^{-k\lambda} s_0^k(z)$, where P_q is the polynomial of degree q given by $P_q(x) := x(x+1)\dots(x+q-1)$, and the symbol of $D(p,q,k)$ is given by

$$\widehat{D(p,q,k)}(z) = P_q(k\lambda)\, z^p \left(\frac{\overline{z}}{1-|z|^2} \right)^q .$$

It follows that $z^p \left(\frac{\overline{z}}{1-|z|^2} \right)^q$ is the symbol of the densely defined operator $\frac{D(p,q,k)}{P_q(k\lambda)}$ on $\mathcal{H}_k$. We can clearly compose such operators since the result of applying the first to a coherent state is a coherent state for a different parameter and these are in the domain of the second.

The disk

So the $*_k$- defined in (**??**) is well-defined on those functions and yields

$$\left\{ z^p \left(\frac{\bar{z}}{1-|z|^2} \right)^q \right\} *_k \left\{ z^r \left(\frac{\bar{z}}{1-|z|^2} \right)^s \right\} = \left(P_q(k\lambda)P_s(k\lambda)\right)^{-1} \widehat{D(p,q,k) \circ D}(r,s,k)$$
$$= \sum_{m=0}^{\min(q,r)} \binom{q}{m} \frac{r!}{(r-m)!} \frac{P_{s+q-m}(k\lambda)}{P_q(k\lambda)P_s(k\lambda)} z^{p+r-m} \left(\frac{\bar{z}}{1-|z|^2} \right)^{s+q-m}.$$

We deduce that $\left\{ z^p \left(\frac{\bar{z}}{1-|z|^2} \right)^q \right\} *_k \left\{ z^r \left(\frac{\bar{z}}{1-|z|^2} \right)^s \right\}$ is a rational function of k; hence the asymptotic expansion is convergent on symbols of polynomial differential operators.

We have on the disk a subspace of smooth functions $\left\{ z^p \left(\frac{\bar{z}}{1-|z|^2} \right)^q \right\}$, with a family of associative products $\{*_k\}$.

The dual of a Lie algebra

The star product on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ obtained via the bijection between polynomials on $\mathfrak{g}^*$ and the universal enveloping algebra, has also an integral formula counterpart for $\nu = 2\pi i$, given by Drinfeld :

$$u * v(\xi) = \int_{\mathfrak{g}\times\mathfrak{g}} \hat{u}(X)\hat{v}(Y)e^{2i\pi\langle\xi, CBH(X,Y)\rangle} dXdY$$

where $\hat{u}(X) = \int_{\mathfrak{g}^*} u(\eta)e^{-2i\pi\langle\eta,X\rangle} d\eta$ and where CBH denotes Campbell-Baker-Hausdorff formula for the product of elements in the group in a logarithmic chart

$$\exp X \exp Y = \exp CBH(X, Y) \quad \forall X, Y \in \mathfrak{g}.$$

So has the star product on the cotangent bundle of a Lie group.

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Definition of strict deformation quantization

In the framework of C^*-algebras, Rieffel introduced the notion of strict deformation quantization: A **strict deformation quantization** of a dense $*$-subalgebra $\mathbb{A}'$ of a C^*-algebra, in the direction of a Poisson bracket $\{.,.\}$ defined on $\mathbb{A}'$, is an open interval $I \subset \mathbb{R}$ containing 0, the assignment, for each $\hbar \in I$, of:

- an associative product $\times_\hbar$,
- an involution $*_\hbar$,
- and a C^*-norm $\| \ \|_\hbar$ (for $\times_\hbar$ and $*_\hbar$) on $\mathbb{A}'$,

which coincide for $\hbar = 0$ to the original product, involution and C^*-norm on $\mathbb{A}'$,
such that the corresponding field of C^*-algebras, with continuity structure given by the elements of $\mathbb{A}'$ as constant fields, is a continuous field of C^*-algebras,
and such that for all $a, b \in \mathbb{A}'$, $\|\frac{(a\times_\hbar b - ab)}{i\hbar} - \{a,b\}\|_\hbar \to 0$ as $\hbar \to 0$.

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Construction of a strict deformation quantization

Group actions appear here in an essential way : Rieffel introduced a general way to construct such C^*-algebraic deformations based on a strongly continuous isometrical action of $\mathbb{R}^d$ on a C^*-algebra $\mathbb{A}$

$$\alpha : \mathbb{R}^d \times \mathbb{A} \to \mathbb{A} : (x, a) \mapsto \alpha_x a.$$

The product formula for the smooth vectors $\mathbb{A}^\infty$ with respect to this action is defined, using an oscillatory integral, choosing a fixed element θ in the orthogonal Lie algebra $so(d)$, by

$$a \times_\hbar b := a *_\theta^\alpha b := (\frac{1}{\pi\hbar})^d \int_{\mathbb{R}^d \times \mathbb{R}^d} \alpha_x(a)\alpha_y(b) \exp^{\frac{2i}{\hbar} x\cdot\theta y} dxdy$$

and it gives a pre C^* associative algebra structure on $\mathbb{A}^\infty$.

This generalizes the Weyl quantization of $\mathbb{R}^{2n}$. Indeed

$$(F *_\hbar G)(u) := (\frac{1}{\pi\hbar})^{2n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{2i}{\hbar}\Omega(v,w)} F(u+v)G(u+w)dvdw$$

can be rewritten as

$$F *_\hbar G = (\frac{1}{\pi\hbar})^{2n} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} \tau_v(F)\tau_w(G)\mathrm{e}^{\frac{2i}{\hbar}\Omega(v,w)} dvdw$$

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Strict deformation quantization

Bieliavsky and Gayral have generalized the construction to actions of Lie groups that admit negatively curved left-invariant Kähler structure.

An important observation due to Weinstein is the relevance in the phase appearing in the product kernel

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of the symplectic flux $S(x, y, z) = \Omega(x, y) + \Omega(y, z) + \Omega(z, x)$ through a geodesic triangle that admits the points x, y and z as mid-points of its geodesic edges.

This lead to the study of symplectic groups which have a structure of symmetric symplectic spaces. Bieliavsky and his collaborators have built,with increasing generality, analogues of Weyl's quantization : they gave universal deformation formulas for those groups and obtained new examples of strict deformation quantization.

A difficulty arising considering convergent star products given by integral formulas (like the Moyal convergent star product defined on the space of Schwartz functions on $\mathbb{R}^{2n}$) is to extend the construction to infinite dimensional cases, and such an extension is necessary to have a deformation quantization approach for quantum field theory.

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Convergent deformation quantization

Another approach to the convergence problem is the following. Taking the formal power series defining a formal star product, one can ask for convergence in a mathematically meaningful way.

This has been achieved by Waldmann et al. in a growing number of examples, for instance the Wick star product on $\mathbb{C}^n$ and even in infinite dimension, the star product obtained by reduction on the disk, the so-called Gutt star product on the dual of a Lie algebra, a Wick type star product on the sphere.

They take a class of functions on which the star product obviously converges, build seminorms which garantee the continuity of the deformed multiplication, and extend the product by continuity to the completion of the class .

In this way, they construct topological non-commutative algebras, over $\mathbb{C}$ and not just over $C[[\nu]]$, essentially of Fréchet type. They study Hilbert space representations of these algebras by a priori unbounded operators .

Convergence of Moyal star product on a Fréchet algebra had also been studied by Omori et al.

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In this way, they construct topological non-commutative algebras, over $\mathbb{C}$ and not just over $C[[\nu]]$, essentially of Fréchet type. They study Hilbert space representations of these algebras by a priori unbounded operators .

Convergence of Moyal star product on a Fréchet algebra had also been studied by Omori et al.

Convergent deformation quantization

Another approach to the convergence problem is the following. Taking the formal power series defining a formal star product, one can ask for convergence in a mathematically meaningful way.

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Thank you for your attention!