Sufficient conditions for embeddings in generalized space forms

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I had the pleasure of talking with Franki Dillen about some ongoing work on symplectic space forms and their submanifolds when we were preparing the PADGE meeting. His untimely death is a great loss and I dedicate to his memory this short note which I hope he would have liked.

Abstract

A large part of the theory of submanifolds of Riemannian manifolds is devoted to the theory of submanifolds in space forms. It is based on two theorems. The first one states that if M is a submanifold of a Riemannian space form $(N, \mathring{g})$, then the curvature of M and the curvature of the normal bundle νM of M are determined in terms of a constant k, the metric $\mathring{g}$ and the second fundamental form α of the submanifold (by Gauss and Ricci's formulas), and this second fundamental form obeys a system of first order differential equations (Codazzi's equations). The second theorem gives a reciprocal of this and states that if (M, g) is a m dimensional Riemannian manifold, if E is a vector bundle of rank p on M, endowed with a Riemannian structure $\tilde{g}$, if $\tilde{\nabla}$ is a metric connection on E, if α is a symmetric 2-form on M with values in E, and if the curvature of M, the curvature of E, and α satisfy some equations, then, for each point $x \in M$, there exists an open neighborhood U of x in M and an isometric immersion f of this neighborhood in a $(m+p)$-dimensional space space form $(N, \mathring{g})$, such that α is the second fundamental form and such that E is isomorphic to the normal bundle. We rewrite Spivak's proof [1] of this theorem in the language of principal bundles, in a framework that encompasses both the pseudo-Riemannian and the symplectic context.

1 The setup of $\mathcal{B}$-structures, and almost $(\mathring{\mathrm{b}}, \mathring{\rho})$-space forms

Definition 1.1. A $\mathcal{B}$-structure on a manifold M is a non degenerate covariant 2-tensor b, which is either symmetric (hence a pseudo-Riemannian structure) or skewsymmetric. A $\mathcal{B}$-connection is a linear connection ∇ on M that is torsion free and preserves b, so $\nabla \mathrm{b} = 0$ and $T^\nabla = 0$. If b is symmetric, $\nabla = \nabla^{LC}$ is the Levi Civita connection of b; if b is skewsymmetric, then $\nabla \mathrm{b} = 0$ and $T^\nabla = 0 \Rightarrow d\mathrm{b} = 0$, so b defines a symplectic structure and ∇ is a symplectic connection. Recall that on any symplectic manifold there exists a symplectic connection but it is not unique. Similarly, a $\mathcal{B}$-structure on a vector bundle E is a smooth non degenerate symmetric or skewsymmetric bilinear form on each fiber.

Definition 1.2. A manifold N with a $\mathcal{B}$-structure $\mathring{\mathrm{b}}$, a $\mathcal{B}$-connection $\mathring{\nabla}$ and a parallel field $\mathring{\rho}$ of endomorphisms of the tangent bundle so that $\mathring{\mathrm{b}}(\mathring{\rho}X, Y) = \pm \mathring{\mathrm{b}}(X, \mathring{\rho}Y)$ is said to be an almost $(\mathring{\mathrm{b}}, \mathring{\rho})$*-space form* if the curvature $R^{\mathring{\nabla}}$ is defined entirely in terms of $\mathring{\mathrm{b}}$ and $\mathring{\rho}$; more precisely if

$$\mathring{\mathrm{b}}\left(R^{\mathring{\nabla}}(U, V)W, T\right) = F_{(\mathring{\mathrm{b}}, \mathring{\rho})}(U, V, W, T) \tag{1}$$

where $F_{(\overset{\circ}{\mathrm{b}},\mathring{\rho})}$ is a polynomial function with constant coefficients in $\overset{\circ}{\mathrm{b}}$ and $\mathring{\rho}$ involving catenation so that the covariant 4-tensor $F_{(\overset{\circ}{\mathrm{b}},\mathring{\rho})}$ vanishes as soon as one of the vectors is $\overset{\circ}{\mathrm{b}}$-orthogonal to the other three. Remark that an almost $(\overset{\circ}{\mathrm{b}},\mathring{\rho})$-space form is automatically locally symmetric and this implies that the connection $\overset{\circ}{\nabla}$ is unique.

Examples of almost $(\overset{\circ}{\mathrm{b}},\mathring{\rho})$-space forms are pseudo-Riemannian space forms (for $\overset{\circ}{\mathrm{b}}=\mathring{g},\mathring{\rho}=0$; indeed $\mathring{g}_x(\mathring{R}_x(X,Y)Z,T)=k\,(\mathring{g}_x(X,Z)\mathring{g}_x(Y,T)-\mathring{g}_x(X,T)\mathring{g}_x(Y,Z)))$, Kähler space forms (for $\overset{\circ}{\mathrm{b}}=\mathring{g},\mathring{\rho}=J$; indeed $\mathring{g}_x(\mathring{R}_x(X,Y)Z,T)=k'\Big(\mathring{g}_x(X,Z)\mathring{g}_x(Y,T)-\mathring{g}_x(X,T)\mathring{g}_x(Y,Z)+\mathring{g}_x(X,JZ)\mathring{g}_x(Y,JT)-\mathring{g}_x(X,JT)\mathring{g}_x(Y,JZ)+2\mathring{g}_x(X,JY)\mathring{g}_x(Z,JT)\Big))$ and symplectic space forms as introduced in [2].
We shall give conditions for a manifold M to embed in a $(\overset{\circ}{\mathrm{b}},\mathring{\rho})$-space form.

2 The tool : extended connections on principal bundles

Let $p^P : P \to M$ be a principal bundle on M, with structure group G and let γ be a connection 1-form on P. Recall that γ is a $\mathfrak{g}$ valued 1-form on P, where $\mathfrak{g}$ is the Lie algebra of G, satisfying $\gamma(A^{*P})=A$ for any $A\in\mathfrak{g}$ and $R_g^*\gamma=\mathrm{Ad}\,g^{-1}\gamma$ for any $g\in G$, with A^{*P} the fundamental vector field on P associated to A, $A^{*P}_{\mathfrak{f}}=\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{f}\exp tA_{|0}$ for any $\mathfrak{f}\in P$.
The curvature β^γ of the connection γ is the $\mathfrak{g}$-valued 2-form $\beta^\gamma=\mathrm{d}\gamma+\frac{1}{2}\left[\gamma\overset{\wedge}{,}\gamma\right]$ defined by

$$\beta^\gamma(Z,Z')=\mathrm{d}\gamma(Z,Z')+[\gamma(Z)\,,\,\gamma(Z')]. \tag{2}$$

When P is a frame bundle of a vector bundle $p^E : E\to M$ with fiber V , i.e.

$$P_x\subset\{\,\mathfrak{f}: V\to E_x \text{ linear isomorphisms}\},\quad G\subset\mathfrak{gl}(V)$$

any section $s\in\Gamma(E)$ is identified with the G-equivariant function $\tilde{s}: P\to V$ defined by $\tilde{s}(\mathfrak{f})=\mathfrak{f}^{-1}s(p^P(\mathfrak{f}))$, any connection 1-form γ on P corresponds to a covariant derivative ∇ of sections of E and vice versa through

$$\widetilde{\nabla^\gamma_X s}(\mathfrak{f})=\overline{X}^\gamma_{\mathfrak{f}}\tilde{s} \tag{3}$$

where $\overline{X}^\gamma_{\mathfrak{f}}$ is the horizontal lift in $\mathrm{T}_{\mathfrak{f}}P$ of $X\in\mathrm{T}_{p^P(\mathfrak{f})}M$ defined by $p^P_*\overline{X}^\gamma_{\mathfrak{f}}=X$ and $\gamma_{\mathfrak{f}}(\overline{X}^\gamma_{\mathfrak{f}})=0$.
The curvature 2-form β^γ is linked to the curvature $R^{\nabla^\gamma}(X,Y):=\left[\nabla^\gamma_X,\nabla^\gamma_Y\right]-\nabla^\gamma_{[X,Y]}$ through

$$\beta^\gamma_{\mathfrak{f}}(Z,Z')=\mathfrak{f}^{-1}\circ R^{\nabla^\gamma}_{p^P\mathfrak{f}}\left(p^P_*Z,p^P_*Z'\right)\circ\mathfrak{f}. \tag{4}$$

Let now G' be a Lie group containing G as a closed subgroup. Suppose we are given a $\mathfrak{g}'$-valued 1-form Γ on$P$ such that:

$$\Gamma(A^{*P})=A\quad\forall A\in\mathfrak{g}\qquad\text{and}\qquad R_g^*\Gamma=\mathrm{Ad}(g^{-1})\Gamma\quad\forall g\in G. \tag{5}$$

From these data, we construct a G' principal bundle $p^{P'} : P'\to M$:

$$P'=P\times_G G'=\{[\mathfrak{f},g']=[\mathfrak{f}g,g^{-1}g']\,|\,\mathfrak{f}\in P,g'\in G',g\in G\}$$
$$p^{P'}: P'\to M:[\mathfrak{f},g']\to p^P(\mathfrak{f})\text{ and the right action of }G'\text{ on }P'\text{ is }[\mathfrak{f},g']\,g'':=[\mathfrak{f},g'g''],$$

and a connection 1-form $\tilde{\Gamma}$ on it. Let $p' : P\times G'\to P' : (\mathfrak{f},g')\mapsto[\mathfrak{f},g']$ be the canonical projection. Then $\ker p'_{*(\mathfrak{f},g')}=\{\,A^{*P}_{\mathfrak{f}}-\hat{A}_{g'}\,|\,A\in\mathfrak{g}\,\}$ where $\hat{A}$ denotes the right invariant vector field on G' corresponding to $A\in\mathfrak{g}\subset\mathfrak{g}'$. Indeed the projection p' is the projection on the orbits for the right action of G on $P\times G'$ defining the equivalence, i.e. given by $((\mathfrak{f},g'),g)\mapsto(\mathfrak{f}g,g^{-1}g')$. Define a 1-form $\check{\Gamma}$ on $P\times G'$ with values in $\mathfrak{g}'$ by:

$$\check{\Gamma}_{(\mathfrak{f},g')}(Z+\hat{B}) := \mathrm{Ad}_{g'^{-1}}(\Gamma_{\mathfrak{f}}(Z)+B) \quad (B\in\mathfrak{g}', X\in \mathrm{T}_{\mathfrak{f}}P). \tag{6}$$

The form $\check{\Gamma}$ vanishes on $\ker p'_*$ and is invariant under the G action defining the equivalence. Hence there exists a $\mathfrak{g}'$-valued 1-form Γ' on P' such that

$$\check{\Gamma} = p'^*\Gamma' \tag{7}$$

The $\mathfrak{g}'$-valued 1-form Γ' on P' defined by (7) is a connection form.
Observe that one has a natural G equivariant injection $j: P\to P' : \mathfrak{f}\mapsto[\mathfrak{f},1]$. Then

$$j^*\Gamma' = \Gamma. \tag{8}$$

Introducing the 2-form $\beta^\Gamma := d\Gamma + \frac{1}{2}\left[\Gamma \overset{\wedge}{,} \Gamma\right]$ on P with values in $\mathfrak{g}'$, we have $\beta^\Gamma = j^*\beta^{\Gamma'}$.
If P' is a frame bundle for a vector bundle $p^E: E\to M$, then P can be seen as a bundle of frames for E; if $\nabla^{\Gamma'}$ denotes the covariant derivative of sections of E associated to Γ' then

$$\beta^\Gamma_{\mathfrak{f}}(Z,Z') = \mathfrak{f}^{-1}\circ R^{\nabla^{\Gamma'}}_{p^P\mathfrak{f}}\left(p^P_*Z, p^P_*Z'\right)\circ\mathfrak{f}. \tag{9}$$

3 Sufficient conditions to have an embedding

We consider a manifold of dimension m with a $\mathcal{B}$-structure (M,b) and a $\mathcal{B}$-connection ∇, and a $\mathcal{B}$-vector bundle of rank e over M, $(E,\tilde{\mathrm{b}})\to M$, with a connection $\tilde{\nabla}$ preserving $\tilde{\mathrm{b}}$. We assume that we are given a symmetric E-valued 2-form α on M : this means that for any $x\in M$ one has a symmetric bilinear map $\alpha_x : \mathrm{T}_xM\times\mathrm{T}_xM\to E_x$ where E_x denotes the fiber of E above x. We define a corresponding map A associating to an element $\xi\in E_x$ the endomorphism A_ξ of T_xM, through

$$\tilde{\mathrm{b}}(\xi,\alpha(X,Y)) = \mathrm{b}(A_\xi(X),Y). \tag{10}$$

We assume that we are given a field ρ_1 of endomorphisms of the tangent bundle $\mathrm{T}M$ so that

$$\mathrm{b}\left(\rho_1 X, Y\right) = \pm\mathrm{b}\left(X,\rho_1 Y\right)) \tag{11}$$

and so that it is covariantly constant $\nabla\rho_1 = 0$, i.e. $\nabla_X(\rho_1 Y) = \rho_1\nabla_X Y$ for all $X,Y\in\mathfrak{X}(M)$.
We also assume that we are given a field ρ_2 of endomorphisms of the bundle E so that

$$\tilde{\mathrm{b}}\left(\rho_2\xi,\eta\right) = \pm\tilde{\mathrm{b}}\left(\xi,\rho_2\eta\right)) \tag{12}$$

(with the same sign $\pm$ as for ρ_1), so that it is covariantly constant: $\tilde{\nabla}\rho_2 = 0$, i.e. $\rho_2\tilde{\nabla}_X\xi = \tilde{\nabla}_X\rho_2\xi$ for all $X\in\mathfrak{X}(M)$ and ξ,η in the space $\Gamma(E)$ of sections of the bundle E. We assume furthermore that

$$\rho_2\alpha(X,Y) = \alpha(X,\rho_1 Y) \qquad \forall X,Y\in\mathfrak{X}(M). \tag{13}$$

Let us remark that we can choose $\rho_1 = 0$ and $\rho_2 = 0$.

We choose a model for the tangent space to M at any point x_0 endowed with the endomorphism $(\rho_1)_{x_0}$ and the 2-form b_{x_0}, so we fix a vector space V_1 of dimension m, a nondegenerate symmetric or skewsymmetric bilinear form B_1 on V_1 and an endomorphism $S_1\in\mathfrak{gl}(V_1)$ and we consider the principal bundle over M of adapted frames of $\mathrm{T}M$, $p^{B(\mathrm{T}M)} : B(\mathrm{T}M,\mathrm{b},\rho_1)\to M$ whose fiber at $x\in M$ consists of all linear isomorphisms

$$\{\,\mathfrak{f}: V_1\to\mathrm{T}_xM \mid \mathfrak{f}^*\mathrm{b}_x = \mathrm{B}_1,\ \mathfrak{f}S_1\mathfrak{f}^{-1} = \rho_{1x}\,\};$$

its structure group is $\mathrm{GL}_{S_1}(V_1,\mathrm{B}_1)$ with $\mathrm{GL}_S(V,\mathrm{B}) := \{\, A \in \mathrm{GL}(V,\mathrm{B}) \,|\, AS = SA \,\}$ and

$$\mathrm{GL}(V,\mathrm{B}) := \{\, A \in \mathrm{GL}(V) \,|\, \mathrm{B}(AX,AY) = \mathrm{B}(X,Y) \,\}.$$

Similarly, we choose a model for the fiber of the bundle E at any point x_0 endowed with the endomorphism $(\rho_2)_{x_0}$ and the 2-form $\tilde{\mathrm{b}}_{x_0}$, so we fix a vector space V_2 of dimension e, a nondegenerate symmetric or skewsymmetric bilinear form B_2 on V_2 and an endomorphism $S_2 \in \mathfrak{gl}(V_2)$ and we consider the $\mathrm{GL}_{S_2}(V_2,\mathrm{B}_2)$-principal bundle over M of adapted frames of E, $p^{B(E)} : B(E,\tilde{\mathrm{b}},\rho_2) \to M$, whose fiber at $x \in M$ consists of all linear isomorphisms

$$\{\, \tilde{\mathfrak{f}} : V_2 \to E_x \,|\, \tilde{\mathfrak{f}}^* \tilde{\mathrm{b}}_x = \mathrm{B}_2,\ \tilde{\mathfrak{f}} S_2 \tilde{\mathfrak{f}}^{-1} = \rho_{2x} \,\}.$$

Since the connection ∇ preserves b and ρ_1, it is defined by a connection 1-form γ^{∇} on $B(\mathrm{T}M,\mathrm{b},\rho_1)$ with values in the Lie algebra $\mathfrak{gl}_{S_1}(V_1,\mathrm{B}_1)$. Similarly, since the connection $\tilde{\nabla}$ preserves $\tilde{\mathrm{b}}$ and ρ_2, it is defined by a connection 1-form $\gamma^{\tilde{\nabla}}$ on $B(E,\tilde{\mathrm{b}},\rho_2)$ with values $\mathfrak{gl}_{S_2}(V_2,\mathrm{B}_2)$.
We consider the $\mathrm{GL}_{S_1}(V_1,\mathrm{B}_1) \times \mathrm{GL}_{S_2}(V_2,\mathrm{B}_2)$-principal bundle defined by the fiber product

$$B(\mathrm{T}M,\mathrm{b},\rho_1) \times_M B(E,\tilde{\mathrm{b}},\rho_2) := \{\, (\mathfrak{f},\tilde{\mathfrak{f}}) \,|\, p^{B(\mathrm{T}M)}(\mathfrak{f}) = p^{B(E)}(\tilde{\mathfrak{f}}) \,\} \tag{14}$$

and denote by p its projection on M, $p(\mathfrak{f},\tilde{\mathfrak{f}}) := p^{B(\mathrm{T}M)}(\mathfrak{f})$. We denote by q_1 (resp. q_2) the projections on $B(\mathrm{T}M,\mathrm{b},\rho_1)$ (resp. $B(E,\tilde{\mathrm{b}},\rho_2)$). Consider the vector space $V := V_1 \oplus V_2$ endowed with the non degenerate two-form $\mathrm{B} := \mathrm{B}_1 \oplus \mathrm{B}_2$ and consider the endomorphism of V preserving B defined by $S := S_1 \oplus S_2$. Clearly $\mathrm{GL}_{S_1}(V_1,\mathrm{B}_1) \times \mathrm{GL}_{S_2}(V_2,\mathrm{B}_2)$ is a closed subgroup of $\mathrm{GL}_S(V,\mathrm{B})$. We define a 1-form Γ on $B(\mathrm{T}M,\mathrm{b},\rho_1) \times_M B(E,\tilde{\mathrm{b}},\rho_2)$ with values in $\mathfrak{gl}(V)$ by:

$$\Gamma_{(\mathfrak{f},\tilde{\mathfrak{f}})}(Z) = \begin{pmatrix} \gamma^{\nabla}_{\mathfrak{f}}(q_{1*}Z) & -\mathfrak{f}^{-1} \circ A_{\tilde{\mathfrak{f}}(\,\cdot\,)} p_* Z \\ \tilde{\mathfrak{f}}^{-1} \circ \alpha(p_*Z, \mathfrak{f}\,\cdot\,) & \gamma^{\tilde{\nabla}}_{\tilde{\mathfrak{f}}}(q_{2*}Z) \end{pmatrix} \tag{15}$$

where $Z \in \mathrm{T}_{(\mathfrak{f},\tilde{\mathfrak{f}})} B(\mathrm{T}M,\mathrm{b},\rho_1) \times_M B(\nu M,\tilde{\mathrm{b}},\rho_2)$. A direct calculation shows:

Lemma 3.1. *Γ has values in the Lie algebra $\mathfrak{gl}_S(V,\mathrm{B})$ and is $G := \mathrm{GL}_{S_1}(V_1,\mathrm{B}_1) \times \mathrm{GL}_{S_2}(V_2,\mathrm{B}_2)$ covariant in the sense that it satisfies equations 5. Hence Γ induces a connection 1-form Γ' on*

$$P' := (B(\mathrm{T}M,\mathrm{b},\rho_1) \times_M B(\nu M,\tilde{\mathrm{b}},\rho_2)) \times_G \mathrm{GL}_S(V,\mathrm{B}).$$

A tangent vector in $\mathrm{T}_{(\mathfrak{f},\tilde{\mathfrak{f}})}\left(B(\mathrm{T}M,\mathrm{b},\rho_1) \times_M B(\nu M,\tilde{\mathrm{b}},\rho_2)\right)$ is given by a pair of vectors $(U \in \mathrm{T}_{\mathfrak{f}} B(\mathrm{T}M,\mathrm{b},\rho_1), V \in \mathrm{T}_{\tilde{\mathfrak{f}}} B(\nu M,\tilde{\mathrm{b}},\rho_2)$ which project on the same tangent vector to M. The horizontal lift for the connection 1-form Γ' of the vector $X \in \mathrm{T}_x M$ in
$\mathrm{T}\left((B(\mathrm{T}M,\mathrm{b},\rho_1) \times_M B(\nu M,\tilde{\mathrm{b}},\rho_2)) \times_G \mathrm{GL}_S(V,\mathrm{B})\right)$ is given at the point $j(\mathfrak{f},\tilde{\mathfrak{f}}) = [(\mathfrak{f},\tilde{\mathfrak{f}},1)]$ by

$$\begin{aligned} \overline{X}^{\Gamma'}_{[(\mathfrak{f},\tilde{\mathfrak{f}},1)]} &= j_{*(\mathfrak{f},\tilde{\mathfrak{f}})}\left(\overline{X}^{\gamma}_{\mathfrak{f}}, \overline{X}^{\tilde{\gamma}}_{\tilde{\mathfrak{f}}}\right) - \left(\left(\Gamma_{(\mathfrak{f},\tilde{\mathfrak{f}})}\left(\overline{X}^{\gamma}_{\mathfrak{f}}, \overline{X}^{\tilde{\gamma}}_{\tilde{\mathfrak{f}}}\right)\right)^{*P'}\right)_{[(\mathfrak{f},\tilde{\mathfrak{f}},1)]} \\ &= j_{*(\mathfrak{f},\tilde{\mathfrak{f}})}\left(\overline{X}^{\gamma}_{\mathfrak{f}}, \overline{X}^{\tilde{\gamma}}_{\tilde{\mathfrak{f}}}\right) - \left(\begin{pmatrix} 0 & -\mathfrak{f}^{-1} A_{\tilde{\mathfrak{f}}(\,\cdot\,)} X_x \\ \tilde{\mathfrak{f}}^{-1} \alpha_x(X_x, \mathfrak{f}\,\cdot\,) & 0 \end{pmatrix}^{*P'}\right)_{[(\mathfrak{f},\tilde{\mathfrak{f}},1)]}. \end{aligned}$$

Remark that $P' = (B(\mathrm{T}M,\mathrm{b},\rho_1) \times_M B(\nu M,\tilde{\mathrm{b}},\rho_2)) \times_G \mathrm{GL}_S(V,\mathrm{B})$ is clearly an adapted frame bundle for $(\mathrm{T}M \oplus E, \mathrm{b} \oplus \tilde{\mathrm{b}}, \rho_1 \oplus \rho_2)$, so that Γ' induces a covariant derivative $\nabla^{\Gamma'}$ of the sections of the bundle $\mathrm{T}M \oplus E$. If $\widetilde{Y+\xi}^{P'}$ is the V-valued $\mathrm{GL}_S(V,\mathrm{B})$-equivariant function on

P' corresponding to the section of $\mathrm{T}M \oplus E$ which is the sum of the vector field Y on M and the section ξ of the bundle E we have

$$\left(C^{*P'}\right)_{\mathfrak{f}'} \widetilde{Y+\xi}^{P'} = -C\left((\widetilde{Y+\xi})^{P'}(\mathfrak{f}')\right) \qquad \forall C \in \mathfrak{gl}_S(V,B)$$

by equivariance and

$$\widetilde{Y+\xi}^{P'}([(\mathfrak{f},\tilde{\mathfrak{f}},1)]) = \begin{pmatrix} \tilde{Y}(\mathfrak{f}) \\ \tilde{\xi}(\tilde{\mathfrak{f}}) \end{pmatrix} \text{ with } \tilde{Y}(\mathfrak{f}) = \mathfrak{f}^{-1}Y_x,\ \tilde{\xi}(\tilde{\mathfrak{f}}) = \tilde{\mathfrak{f}}^{-1}\xi_x,\ x = p(\mathfrak{f},\tilde{\mathfrak{f}}),$$

so that

$$\begin{aligned} \nabla^{\Gamma'}\widetilde{(Y+\xi)}([(\mathfrak{f},\tilde{\mathfrak{f}},1)]) &= \overline{X}^{\Gamma'}_{[(\mathfrak{f},\tilde{\mathfrak{f}},1)]}\widetilde{Y+\xi} = \begin{pmatrix} \overline{X}^{\gamma}_{\mathfrak{f}}\tilde{Y} \\ \overline{X}^{\tilde{\gamma}}_{\tilde{\mathfrak{f}}}\tilde{\xi} \end{pmatrix} + \begin{pmatrix} 0 & -\mathfrak{f}^{-1}A_{\tilde{\mathfrak{f}}(\,\cdot\,)}X_x \\ \tilde{\mathfrak{f}}^{-1}\alpha_x(X_x,\mathfrak{f}\,\cdot\,) & 0 \end{pmatrix}\begin{pmatrix} \tilde{Y}(\mathfrak{f}) \\ \tilde{\xi}(\tilde{\mathfrak{f}}) \end{pmatrix} \\ &= \begin{pmatrix} \overline{X}^{\gamma}_{\mathfrak{f}}\tilde{Y} - \mathfrak{f}^{-1}A_{\tilde{\mathfrak{f}}(\tilde{\xi}(\tilde{\mathfrak{f}}))}X_x) \\ \overline{X}^{\tilde{\gamma}}_{\tilde{\mathfrak{f}}}\tilde{\xi} + \tilde{\mathfrak{f}}^{-1}\alpha_x(X_x,\mathfrak{f}\tilde{Y}(\mathfrak{f})) \end{pmatrix} = \begin{pmatrix} \widetilde{\nabla^{\gamma}_X Y}(\mathfrak{f}) - \mathfrak{f}^{-1}A_{\xi_x}X_x) \\ \widetilde{\nabla^{\tilde{\gamma}}_X\xi}(\tilde{\mathfrak{f}}) + \tilde{\mathfrak{f}}^{-1}\alpha_x(X_x,Y_x) \end{pmatrix}; \end{aligned}$$

hence the covariant derivative $\nabla^{\Gamma'}$ is defined by

$$\begin{aligned} \nabla^{\Gamma'}_X Y &= \nabla_X Y + \alpha(X,Y) & (16) \\ \nabla^{\Gamma'}_X \xi &= -A_\xi X + \tilde{\nabla}_X \xi. & (17) \end{aligned}$$

for $X, Y \in \mathfrak{X}(M), \xi \in \Gamma(E)$ and we have as in (9) that

$$\beta^{\Gamma}_{(\mathfrak{f},\tilde{\mathfrak{f}})}(Z,Z') = (\mathfrak{f},\tilde{\mathfrak{f}})^{-1} \circ R^{\nabla^{\Gamma'}}_{p(\mathfrak{f},\tilde{\mathfrak{f}})}\left(p_*Z, p_*Z'\right) \circ (\mathfrak{f},\tilde{\mathfrak{f}}) \tag{18}$$

for $\beta^{\Gamma} = d\Gamma + \frac{1}{2}\left[\Gamma \overset{\wedge}{,} \Gamma\right]$. By direct calculation, the curvature of $\nabla^{\Gamma'}$ writes

$$\begin{aligned} R^{\nabla^{\Gamma'}}(X,Y)Z &= R^{\nabla}(X,Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X + (D^{\nabla\tilde{\nabla}}_X\alpha)(Y,Z) - (D^{\nabla\tilde{\nabla}}_Y\alpha)(X,Z) \\ R^{\nabla^{\Gamma'}}(X,Y)\xi &= R^{\tilde{\nabla}}(X,Y)\xi + \alpha(Y,A_\xi X) - \alpha(X,A_\xi Y) + (D^{\nabla\tilde{\nabla}}_Y A)_\xi X - (D^{\nabla\tilde{\nabla}}_X A)_\xi Y. & (19) \end{aligned}$$

We now introduce the so called "soldering" forms defined on any bundle of frames of the tangent bundle of a manifold, in particular on $B(\mathrm{T}M, \mathrm{b}, \rho_1)$.

Definition 3.1. The *soldering form* $\theta^{B(\mathrm{T}M)}$ is the 1-form on $B(\mathrm{T}M, \mathrm{b}, \rho_1)$ with values in V_1 given by

$$\theta^{B(\mathrm{T}M)}_{\mathfrak{f}}(Z) = \mathfrak{f}^{-1}(p_*^{B(\mathrm{T}M,\mathrm{b},\rho_1)}Z) \tag{20}$$

for any $Z \in \mathrm{T}_{\mathfrak{f}}B(\mathrm{T}M, \mathrm{b}, \rho_1)$.

Lemma 3.2. *Let γ^{∇} be a connection 1-form on the frame bundle so that the induced linear connection is without torsion. Then the differential of the soldering form $\theta^{B(\mathrm{T}M)}$ is the V_1-valued 2-form on $B(\mathrm{T}M, \mathrm{b}, \rho_1)$ given by*

$$\mathrm{d}\theta^{B(\mathrm{T}M)}_{\mathfrak{f}}(Z,Z') = -\gamma^{\nabla}_{\mathfrak{f}}(Z)\left(\theta^{B(\mathrm{T}M)}_{\mathfrak{f}}(Z')\right) + \gamma^{\nabla}_{\mathfrak{f}}(Z')\left(\theta^{B(\mathrm{T}M)}_{\mathfrak{f}}(Z)\right), \tag{21}$$

which we write as $\mathrm{d}\theta^{B(\mathrm{T}M)} = -\gamma^{\nabla}_{\mathfrak{f}} \overset{\wedge}{\cdot} \theta^{B(\mathrm{T}M)}$.

The formula is classical and easily checked using horizontal lifts $\bar{Y}$ of vector fields Y on M and fundamental f (vertical) vector fields A^* with $A \in \mathfrak{gl}_{S_1}(V_1, \mathrm{b}_1)$.

Proposition 3.1. *Let $M, \mathrm{b}, \nabla, \rho_1, E, \tilde{\mathrm{b}}, \tilde{\nabla}, \rho_2, \alpha, V_1, \mathrm{B}_1, S_1, V_2, \mathrm{B}_2, S_2$ be defined as before. Let N be a manifold of dimension $n := m+e$ endowed with a $\mathcal{B}$-structure $\mathring{\mathrm{b}}$, a connexion $\mathring{\nabla}$ preserving $\mathring{\mathrm{b}}$, and a parallel field of endomorphism $\mathring{\rho}$ so that $(V = V_1 \oplus V_2, S = S_1 \oplus S_2, \mathrm{B} = \mathrm{B}_1 \oplus \mathrm{B}_2)$ is a model for $(T_{y_0}N, (\mathring{\rho})_{y_0}, \mathring{\mathrm{b}}_{y_0})$ at any point y_0. Assume that $(N, \mathring{\mathrm{b}}, \mathring{\nabla}, \mathring{\rho})$ is an almost $(\mathring{\mathrm{b}}, \mathring{\rho})$-space form with $F_{(\mathring{\mathrm{b}},\mathring{\rho})}$ defined as in (1). Define $\widehat{R^{\mathring{\nabla}}}_{\mathrm{b},\tilde{\mathrm{b}},\rho_1,\rho_2}$, a covariant 2-tensor on M with values in the endomorphisms of $\mathrm{T}M \oplus E = \mathrm{T}M \times_M E$, by*

$$(\mathrm{b} \oplus \tilde{\mathrm{b}}) \left(\widehat{R^{\mathring{\nabla}}}_{\mathrm{b},\tilde{\mathrm{b}},\rho_1,\rho_2}(X,Y)(Z+\xi), Z'+\xi' \right) = F_{(\mathrm{b}\oplus\tilde{\mathrm{b}},\rho_1\oplus\rho_2)}(X+0, Y+0, Z+\xi, Z'+\xi'). \quad (22)$$

Consider the principal bundle of adapted frames of $\mathrm{T}N$, $p^{B(\mathrm{T}N)} : B(\mathrm{T}N, \mathring{\mathrm{b}}, \mathring{\rho}) \to N$; its fiber at $y \in N$ consists of all linear isomorphisms $\{ \mathring{\mathfrak{f}} : V \to \mathrm{T}_y N \,|\, \mathring{\mathfrak{f}}^* \mathring{\mathrm{b}} = \mathrm{B},\ \mathring{\mathfrak{f}} S \mathring{\mathfrak{f}}^{-1} = \mathring{\rho}_y \}$; its structure group is $\mathrm{GL}_S(V, \mathrm{B})$. Let B be the bundle over $M \times N$ defined by

$$B := \Big(B(\mathrm{T}M, \mathrm{b}, \rho_1) \times_M B(E, \tilde{\mathrm{b}}, \rho_2) \Big) \times B(\mathrm{T}N, \mathring{\mathrm{b}}, \mathring{\rho});$$

denote by p_1 its projection on $B(\mathrm{T}M, \mathrm{b}, \rho_1) \times_M B(E, \tilde{\mathrm{b}}, \rho_2)$ and p_2 its projection on $B(\mathrm{T}N, \mathring{\mathrm{b}}, \mathring{\rho})$. Define the distribution $\mathcal{D}$ on B as the kernel of the family of 1-forms on B given by

$$p_1^*\Gamma - p_2^*\gamma^{\mathring{\nabla}} \quad \text{with } \Gamma \text{ as defined in (15); those forms have values in } \mathfrak{gl}_S(V,\mathrm{B}), \quad (23)$$
$$p_1^* q_1^* \theta^{B(\mathrm{T}M)} - p_2^* \theta^{B(\mathrm{T}N)} \quad \text{with values in } V \supset V_1. \quad (24)$$

This distribution is integrable if the following equations are satisfied :

$$\widehat{R^{\mathring{\nabla}}}_{\mathrm{b},\tilde{\mathrm{b}},\rho,\tilde{\rho}}(X,Y)Z = R^{\nabla}(X,Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X \qquad \text{(Gauss's formula)} \quad (25)$$
$$0 = (D_X^{\nabla\tilde{\nabla}}\alpha)(Y,Z) - (D_Y^{\nabla\tilde{\nabla}}\alpha)(X,Z) \qquad \text{(Codazzi's formula)} \quad (26)$$
$$\widehat{R^{\mathring{\nabla}}}_{\mathrm{b},\tilde{\mathrm{b}},\rho,\tilde{\rho}}(X,Y)\xi = R^{\tilde{\nabla}}(X,Y)\xi + \alpha(Y, A_\xi X) - \alpha(X, A_\xi Y) \qquad \text{(Ricci's formula)} \quad (27)$$
$$\text{for any } X, Y, Z \in \mathfrak{X}(M), \xi \in \Gamma(E).$$

Proof. To show the integrability of the distribution, we shall show that the differential of the 1-forms defining the distribution belong to the ideal generated by those 1-forms. We have seen in lemma 3.2 that $d\theta = -\gamma^{\nabla} \overset{\wedge}{\cdot} \theta$ so that

$$\mathrm{d}\left(p_1^* q_1^* \theta^{B(\mathrm{T}M)} - p_2^* \theta^{B(\mathrm{T}N)} \right) = -p_1^* q_1^* \left(\gamma^{\nabla} \overset{\wedge}{\cdot} \theta^{B(\mathrm{T}M)} \right) + p_2^* \left(\gamma^{\mathring{\nabla}} \overset{\wedge}{\cdot} \theta^{B(\mathrm{T}N)} \right)$$

On the other hand

$$\left((\Gamma - q_1^*\gamma^{\nabla}) \overset{\wedge}{\cdot} q_1^* \theta^{B(\mathrm{T}M)} \right)_{(\mathfrak{f},\tilde{\mathfrak{f}})} (U,V) = \tilde{\mathfrak{f}}^{-1} \alpha \left(p_* U , \mathfrak{f} \left(\theta_{\mathfrak{f}}^{B(\mathrm{T}M)}(q_{1*} V) \right) \right)$$
$$- \tilde{\mathfrak{f}}^{-1} \alpha \left(p_* V , \mathfrak{f} \left(\theta_{\mathfrak{f}}^{B(\mathrm{T}M)}(q_{1*} U) \right) \right) = \tilde{\mathfrak{f}}^{-1} \alpha(p_* U, p_* V) - \tilde{\mathfrak{f}}^{-1} \alpha(p_* V, p_* U) = 0$$

since α is symmetric. Hence

$$\mathrm{d}\left(p_1^* q_1^* \theta^{B(\mathrm{T}M)} - p_2^* \theta^{B(\mathrm{T}N)} \right) = -p_1^* \left(\Gamma \overset{\wedge}{\cdot} q_1^* \theta^{B(\mathrm{T}M)} \right) + p_2^* \left(\gamma^{\mathring{\nabla}} \overset{\wedge}{\cdot} \theta^{B(\mathrm{T}N)} \right)$$
$$= - \left(p_1^* \Gamma - p_2^* \gamma^{\mathring{\nabla}} \right) \overset{\wedge}{\cdot} p_1^* q_1^* \theta^{B(\mathrm{T}M)} - p_2^* \gamma^{\mathring{\nabla}} \overset{\wedge}{\cdot} \left(p_1^* q_1^* \theta^{B(\mathrm{T}M)} - p_2^* \theta^{B(\mathrm{T}N)} \right)$$

is in the ideal generated by the 1-forms. On the other hand, we have, as in equation (2), $\beta^{\gamma} = \mathrm{d}\gamma + \frac{1}{2} \left[\gamma \overset{\wedge}{,} \gamma \right]$ and by definition $\beta^{\Gamma} = d\Gamma + \frac{1}{2} \left[\Gamma \overset{\wedge}{,} \Gamma \right]$ so that

$$\mathrm{d}\left(p_1^* \Gamma - p_2^* \gamma^{\mathring{\nabla}} \right) = p_1^* \beta^{\Gamma} - p_2^* \beta^{\gamma^{\mathring{\nabla}}} - \tfrac{1}{2} \left[p_1^* \Gamma \overset{\wedge}{,} p_1^* \Gamma \right] + \tfrac{1}{2} \left[p_2^* \gamma^{\mathring{\nabla}} \overset{\wedge}{,} p_2^* \gamma^{\mathring{\nabla}} \right]$$
$$= p_1^* \beta^{\Gamma} - p_2^* \beta^{\gamma^{\mathring{\nabla}}} - \tfrac{1}{2} \left[p_1^* \Gamma - p_2^* \gamma^{\mathring{\nabla}} \overset{\wedge}{,} p_1^* \Gamma \right] - \tfrac{1}{2} \left[p_2^* \gamma^{\mathring{\nabla}} \overset{\wedge}{,} p_1^* \Gamma - p_2^* \gamma^{\mathring{\nabla}} \right]$$

and it is in the ideal generated by the 1-forms iff $p_1^* \beta^{\Gamma} - p_2^* \beta^{\gamma^{\mathring{\nabla}}}$ is in the ideal. Observe that

$$\beta_{\mathring{\mathfrak{f}}}^{\gamma^{\mathring{\nabla}}}(W, W') = \mathring{\mathfrak{f}}^{-1} \circ R^{\mathring{\nabla}}(p_*^{B(\mathrm{T}N)} W, p_*^{B(\mathrm{T}N)} W') \circ \mathring{\mathfrak{f}}. \quad (28)$$

We write $\mathring{\mathrm{b}}\left(R^{\mathring{\nabla}}(U,V)W,T\right) = F_{(\mathring{\mathrm{b}},\mathring{\rho})}(U,V,W,T)$ in an equivalent way as

$$R^{\mathring{\nabla}}(U,V) = G_{(\mathring{\mathrm{b}},\mathring{\rho})}(U,V) \tag{29}$$

where $G_{(\mathring{\mathrm{b}},\mathring{\rho})}$ is a constant coefficient polynomial in the field of endomorphisms $\mathring{\rho}$ and in the field of 2-tensors $\mathring{\mathrm{b}}$ (involving concatenations) defining a 2-form with values in the endomorphisms. Then

$$\begin{aligned} \beta^{\gamma^{\mathring{\nabla}}}_{\mathring{\mathfrak{f}}}(W,W') &= \mathring{\mathfrak{f}}^{-1}\circ G_{(\mathring{\mathrm{b}},\mathring{\rho})}(\mathring{\mathfrak{f}}\theta^{B(\mathrm{T}N)}W, \mathring{\mathfrak{f}}\theta^{B(\mathrm{T}N)}W')\circ\mathring{\mathfrak{f}} \\ &= G_{(B,S)}(\theta^{B(\mathrm{T}N)}W,\theta^{B(\mathrm{T}N)}W') =: \tfrac{1}{2}G_{(B,S)}\left(\theta^{B(\mathrm{T}N)} \underset{,}{\wedge} \theta^{B(\mathrm{T}N)}\right)(W,W') \end{aligned}$$

and $G_{(B,S)}$ is a 2-form on V with values in $\mathrm{End}(V)$. Similarly, as in (18)

$$\beta^{\Gamma}_{(\mathfrak{f},\tilde{\mathfrak{f}})}(Z,Z') = (\mathfrak{f},\tilde{\mathfrak{f}})^{-1}\circ R^{\nabla^{\Gamma'}}_{p(\mathfrak{f},\tilde{\mathfrak{f}})}(p_*Z,p_*Z')\circ(\mathfrak{f},\tilde{\mathfrak{f}}).$$

where $\nabla^{\Gamma'}$ is the covariant derivative of sections of $\mathrm{T}M\oplus E$ defined as in (16). The Gauss, Codazzi and Ricci formulas imply, in view of (19) that

$$R^{\nabla^{\Gamma'}}(X,Y) = \widehat{R^{\mathring{\nabla}}}_{\mathrm{b},\tilde{\mathrm{b}},\rho_1,\rho_2}(X,Y)\,.$$

and (22) implies that

$$\widehat{R^{\mathring{\nabla}}}_{\mathrm{b},\tilde{\mathrm{b}},\rho_1,\rho_2}(X,Y) = G_{\mathrm{b}\oplus\tilde{\mathrm{b}},\rho_1\oplus\rho_2}(X,Y)$$

so that

$$\begin{aligned} \beta^{\Gamma}_{(\mathfrak{f},\tilde{\mathfrak{f}})}(Z,Z') &= (\mathfrak{f},\tilde{\mathfrak{f}})^{-1}G_{\mathrm{b}\oplus\tilde{\mathrm{b}},\rho_1\oplus\rho_2}(p_*Z,p_*Z')\circ(\mathfrak{f},\tilde{\mathfrak{f}}) \\ &= (\mathfrak{f},\tilde{\mathfrak{f}})^{-1}G_{\mathrm{b}\oplus\mathrm{b},\rho_1\oplus\rho_2}\left(\mathfrak{f}\theta^{B(\mathrm{T}M)}q_{1*}Z,\mathfrak{f}\theta^{B(\mathrm{T}M)}q_{1*}Z'\right)\circ(\mathfrak{f},\tilde{\mathfrak{f}}) \\ &= G_{(B,S)}(\theta^{B(\mathrm{T}M)}q_{1*}Z,\theta^{B(\mathrm{T}M)}q_{1*}Z') = \tfrac{1}{2}G_{(B,S)}\left(q_1^*\theta^{B(\mathrm{T}M)} \underset{,}{\wedge} q_1^*\theta^{B(\mathrm{T}M)}\right)(Z,Z')\,. \end{aligned}$$

Hence

$$\begin{aligned} p_1^*\beta^{\Gamma} - p_2^*\beta^{\gamma^{\mathring{\nabla}}} &= \tfrac{1}{2}G_{(B,S)}\left(p_1^*q_1^*\theta^{B(\mathrm{T}M)} \underset{,}{\wedge} p_1^*q_1^*\theta^{B(\mathrm{T}M)}\right) - \tfrac{1}{2}G_{(B,S)}\left(p_2^*\theta^{B(\mathrm{T}N)} \underset{,}{\wedge} p_2^*\theta^{B(\mathrm{T}N)}\right) \\ &= \tfrac{1}{2}G_{(B,S)}\left(\left(p_1^*q_1^*\theta^{B(\mathrm{T}M)} - p_2^*\theta^{B(\mathrm{T}N)}\right) \underset{,}{\wedge} p_1^*q_1^*\theta^{B(\mathrm{T}M)}\right) \\ &\quad + \tfrac{1}{2}G_{(B,S)}\left(p_2^*\theta^{B(\mathrm{T}N)} \underset{,}{\wedge} \left(p_1^*q_1^*\theta^{B(\mathrm{T}M)} - p_2^*\theta^{B(\mathrm{T}N)}\right)\right) \end{aligned}$$

is again in the ideal. □

Remark that the projection $p_1 : B \to B(\mathrm{T}M,\mathrm{b},\rho_1)\times_M B(E,\tilde{\mathrm{b}},\rho_2)$ induces a smooth local diffeomorphism from any leaf $\mathcal{F}$ to $B(\mathrm{T}M,\mathrm{b},\rho_1)\times_M B(E,\tilde{\mathrm{b}},\rho_2)$. Indeed the differential $p_{1*_{(\mathfrak{f},\tilde{\mathfrak{f}},\mathring{\mathfrak{f}})}}$ is an isomorphism since any tangent vector to B at the point $(\mathfrak{f},\tilde{\mathfrak{f}},\mathring{\mathfrak{f}})$ over $(x,y)\in M\times N$ can be written uniquely as

$$\left(\overline{X}^{\gamma}_{\mathfrak{f}} + (C_1)^{*B(\mathrm{T}M)}_{\mathfrak{f}}, \overline{X}^{\tilde{\gamma}}_{\tilde{\mathfrak{f}}} + (C_2)^{*B(E)}_{\tilde{\mathfrak{f}}}\right) + (\overline{Y}^{\gamma}_{\mathring{\mathfrak{f}}} + C^{*B(\mathrm{T}N)}_{\mathring{\mathfrak{f}}})$$

for $X\in\mathrm{T}_xM$, $Y\in\mathrm{T}_yN$, $C_1\in\mathfrak{gl}_{S_1}(V_1,B_1)$, $C_2\in\mathfrak{gl}_{S_2}(V_2,B_2)$, $C\in\mathfrak{gl}_S(V,B)$, and it belongs to the distribution $\mathcal{D}_{(\mathfrak{f},\tilde{\mathfrak{f}},\mathring{\mathfrak{f}})}$ iff

$$Y_y = \mathring{\mathfrak{f}}\left(\mathfrak{f}^{-1}(X_x)\right) \qquad C = \begin{pmatrix} C_1 & -\mathfrak{f}^{-1}A_{\tilde{\mathfrak{f}}(\,\cdot\,)}X_x \\ \tilde{\mathfrak{f}}^{-1}\alpha_x(X_x,\mathfrak{f}\,\cdot\,) & C_2 \end{pmatrix}. \tag{30}$$

Hence the leaf $\mathcal{F}$ through $(\mathfrak{f},\tilde{\mathfrak{f}},\mathring{\mathfrak{f}})$ can be seen locally as the graph of a smooth immersion

$$\tilde{j}:\tilde{U}\to B(\mathrm{T}N,\mathring{\mathrm{b}},\mathring{\rho})$$

where $\tilde{U}$ is an open neighborhood of $(\mathfrak{f},\tilde{\mathfrak{f}})$ in $B(\mathrm{T}M,\mathrm{b},\rho_1)\times_M B(\nu M,\tilde{\mathrm{b}},\rho_2)$ mapping $(\mathfrak{f},\tilde{\mathfrak{f}})$ to $\mathring{\mathfrak{f}}$. Since, by (30), $\left((C_1)^{*B(\mathrm{T}M)}_{\mathfrak{f}}, (C_2)^{*B(E)}_{\tilde{\mathfrak{f}}}\right) + \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}^{*B(\mathrm{T}N)}_{\mathring{\mathfrak{f}}}$ is in the distribution for all C_1, C_2

and all $\mathfrak{f}, \tilde{\mathfrak{f}}, \mathring{\mathfrak{f}}$, the map $\tilde{j}$ intertwines the local actions of an open neighborhood of $\mathrm{GL}_{S_1}(V_1, B_1) \times \mathrm{GL}_{S_2}(V_2, B_2)$ so that it induces a smooth map

$$j : U \to N$$

where $U = p(\tilde{U})$ is an open neighborhood of x in M. By (30), the differential of j at any point $x \in U$ is given by

$$j_{*x}(X_x) = \mathring{\mathfrak{f}}\left(\mathfrak{f}^{-1}(X_x)\right)$$

where $p(\mathfrak{f}, \tilde{\mathfrak{f}}) = x$ and $\tilde{j}(\mathfrak{f}, \tilde{\mathfrak{f}}) = \mathring{\mathfrak{f}}$ so that

$$j_{*x}(\mathfrak{f}(u)) = \left(\tilde{j}(\mathfrak{f}, \tilde{\mathfrak{f}})\right)(u, 0) \quad \forall u \in V_1.$$

Hence the smooth map

$$\hat{j} : (\mathrm{T}M \oplus E)_{|U} \to \mathrm{T}N : \left(X_x = [(\mathfrak{f}, u)], E_x = [(\tilde{\mathfrak{f}}, v)]\right) \mapsto [\tilde{j}(\mathfrak{f}, \tilde{\mathfrak{f}}), (u, v)]$$

identifies $\mathrm{T}M \oplus E$ to the pullback bundle $j^*\mathrm{T}N$. Thus we have proven

Theorem 3.1. *Let $(M, \mathrm{b}, \nabla, \rho_1)$ be a smooth manifold M of dimension m with a $\mathcal{B}$-structure b, a torsionfree connexion ∇ preserving b, and a parallel field of endomorphism ρ_1 so that $\mathrm{b}(\rho_1 X, Y) = \pm \mathrm{b}(X, \rho_1 Y))$ and let (V_1, S_1, B_1) be a model for $(T_{x_0}M, \rho_{1x_0}, \mathrm{b}_{x_0})$ at a point x_0. Let $(E, \tilde{\mathrm{b}}, \tilde{\nabla}, \rho_2)$ be a vector bundle of rank e over M with a $\mathcal{B}$-structure $\tilde{\mathrm{b}}$, a connexion $\tilde{\nabla}$ preserving $\tilde{\mathrm{b}}$, and a parallel field of endomorphism ρ_2 so that $\tilde{\mathrm{b}}(\rho_2 \xi, \eta) = \pm \tilde{\mathrm{b}}(\xi, \rho_2 \eta))$ (same sign as for ρ_1) and let (V_2, S_2, B_2) be a model for the fiber $(E_{x_0}, \rho_{2x_0}, \tilde{\mathrm{b}}_{x_0})$ at x_0. Let $\alpha : \mathrm{T}M \times \mathrm{T}M \to E$ be a symmetric 2-form on M with values in the endomorphisms of E. Assume that $\rho_2 \alpha(X, Y) = \alpha(X, \rho_1 Y)$ for all $X, Y \in \mathfrak{X}(M)$. Consider a manifold N of dimension $m + e$ endowed with a $\mathcal{B}$-structure $\mathring{\mathrm{b}}$, a connexion $\mathring{\nabla}$ preserving $\mathring{\mathrm{b}}$, and a parallel field of endomorphism $\mathring{\rho}$ so that $(V = V_1 \oplus V_2, S = S_1 \oplus S_2, \mathrm{B} = \mathrm{B}_1 \oplus \mathrm{B}_2)$ is a model for $(T_{y_0}N, (\mathring{\rho})_{y_0}, \mathring{\mathrm{b}}_{y_0})$, and so that N is an almost $(\mathring{\mathrm{b}}, \mathring{\rho})$-space form. Write (1) equivalently as $R^{\mathring{\nabla}}(U, V)W = G_{(\mathring{\mathrm{b}}, \mathring{\rho})}(U, V, W)$. Define the covariant 2-tensor $\widehat{R^{\mathring{\nabla}}}$ on M with values in the endomorphisms of the vector bundle $\mathrm{T}M \oplus E = \mathrm{T}M \times_M E$, by*

$$\widehat{R^{\mathring{\nabla}}}(X, Y)(Z + \xi) = G_{(\mathrm{b} \oplus \tilde{\mathrm{b}}, \rho_1 \oplus \rho_2)}(X, Y, Z + \xi).$$

Assume that we have the Gauss, Codazzi and Ricci equations:

$$\widehat{R^{\mathring{\nabla}}}_{\mathrm{b},,\rho \oplus \tilde{\rho}}(X, Y)Z = R^{\nabla}(X, Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X \quad \text{(Gauss's formula)} \tag{31}$$

with A defined as in (10)

$$0 = (D_X^{\nabla \tilde{\nabla}} \alpha)(Y, Z) - (D_Y^{\nabla \tilde{\nabla}} \alpha)(X, Z) \quad \text{(Codazzi's formula)} \tag{32}$$

$$\widehat{R^{\mathring{\nabla}}}_{\mathrm{b}, \tilde{\mathrm{b}}, \rho, \tilde{\rho}}(X, Y)\xi = R^{\tilde{\nabla}}(X, Y)\xi + \alpha(Y, A_\xi X) - \alpha(X, A_\xi Y) \quad \text{(Ricci's formula)} \tag{33}$$

for any $X, Y, Z \in \mathfrak{X}(M), \xi \in \Gamma(E)$, $X, Y \in \mathfrak{X}(M)$.

Then, around any point x in M, there exists an immersion $j : U \to N$ of an open neighborhhood U of x in N so that $j^*\mathring{\mathrm{b}} = \mathrm{b}$ and so that E is identified to the normal bundle and α to the second fundamental form.

References

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