Involutions and Representations for Reduced Quantum Algebras

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Abstract

In the context of deformation quantization, there exist various procedures to deal with the quantization of a reduced space M_{red}. We shall be concerned here mainly with the classical Marsden-Weinstein reduction, assuming that we have a proper action of a Lie group G on a Poisson manifold M, with a moment map J for which zero is a regular value. For the quantization, we follow [6] (with a simplified approach) and build a star product $\star_{\mathrm{red}}$ on M_{red} from a strongly invariant star product $\star$ on M. The new questions which are addressed in this paper concern the existence of natural *-involutions on the reduced quantum algebra and the representation theory for such a reduced *-algebra.

We assume that $\star$ is Hermitian and we show that the choice of a formal series of smooth densities on the embedded coisotropic submanifold $C = J^{-1}(0)$, with some equivariance property, defines a *-involution for $\star_{\mathrm{red}}$ on the reduced space. Looking into the question whether the corresponding *-involution is the complex conjugation (which is a *-involution in the Marsden-Weinstein context) yields a new notion of quantized unimodular class.

We introduce a left $(\mathcal{C}^\infty(M)[[\lambda]], \star)$-submodule and a right $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$-submodule $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ of $C^\infty(C)[[\lambda]]$; we define on it a $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued inner product and we establish that this gives a strong Morita equivalence bimodule between $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ and the finite rank operators on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$. The crucial point is here to show the complete positivity of the inner product. We obtain a Rieffel induction functor from the strongly non-degenerate *-representations of $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$ on pre-Hilbert right $\mathcal{D}$-modules to those of $(\mathcal{C}^\infty(M)[[\lambda]], \star)$, for any auxiliary coefficient *-algebra $\mathcal{D}$ over $\mathbb{C}[[\lambda]]$.

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Contents

1 Introduction

Some mathematical formulations of quantizations are based on the algebra of observables and consist in replacing the classical algebra of observables $\mathcal{A}$ (typically complex-valued smooth functions on a Poisson manifold M) by a non commutative one $\boldsymbol{\mathcal{A}}$. Formal deformation quantization was introduced in [4]; it constructs the quantum observable algebra by means of a formal deformation (in the sense of Gerstenhaber) of the classical algebra. Given a Poisson manifold M and the classical algebra $\mathcal{A} = C^\infty(M)$ of complex-valued smooth functions, a star product on M is a $\mathbb{C}[[\lambda]]$-bilinear associative multiplication on $\mathcal{C}^\infty(M)[[\lambda]]$ with

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g), \tag{1}$$

where $C_0(f,g) = fg$ and $C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}$, where the C_r are bidifferential operators so that $1 \star f = f = f \star 1$ for all $f \in \mathcal{C}^\infty(M)[[\lambda]]$. The algebra of quantum observables is $\boldsymbol{\mathcal{A}} = (\mathcal{C}^\infty(M)[[\lambda]], \star)$.

An important classical tool to "reduce the number of variables", i.e. to start from a "big" Poisson manifold M and construct a smaller one M_{red}, is given by reduction: one considers an embedded coisotropic submanifold in the Poisson manifold, $\iota : C \hookrightarrow M$ and the canonical foliation of C which we assume to have a nice leaf space M_{red}. In this case one knows that M_{red} is a Poisson manifold in a canonical way.

We shall consider here the particular case of the Marsden-Weinstein reduction: let $\mathsf{L} : G \times M \longrightarrow M$ be a smooth left action of a connected Lie group G on M by Poisson diffeomorphisms and assume we have an ad^*-equivariant momentum map. The constraint manifold C is now chosen to be the level surface of J for momentum $0 \in \mathfrak{g}^*$ (thus we assume, for simplicity, that 0 is a regular value). Then $C = J^{-1}(\{0\})$ is an embedded submanifold which is coisotropic. The group G acts on C and the reduced space is the orbit space of this group action of G on C (in order to guarantee a good quotient we assume that G acts freely and properly).

Given a mathematical formulation of quantization, one studies then a quantized version of reduction and how "quantization commutes with reduction". This has been done in the framework of deformation quantization by various authors [6,12,13]. We shall use here the approach proposed by Bordemann [5] . Since the emphasis is put in our quantization scheme on the observable algebra, recall that at the classical level if $\iota : C \hookrightarrow M$ is an embedded coisotropic submanifold, one considers $\mathcal{J}_C = \{f \in \mathcal{C}^\infty(M) \mid \iota^* f = 0\} = \ker \iota^*$ the vanishing ideal of C [which is an ideal in the associative algebra $\mathcal{C}^\infty(M)$ and a Poisson subalgebra of $\mathcal{C}^\infty(M)$], defining $\mathcal{B}_C = \{f \in \mathcal{C}^\infty(M) \mid \{f, \mathcal{J}_C\} \subseteq \mathcal{J}_C\}$, and assuming that the canonical foliation of C has a nice leaf space M_{red} (i.e. a structure of a smooth manifold such that the canonical projection $\pi : C \longrightarrow M_{\mathrm{red}}$ is a submersion); then

$$\mathcal{B}_C/\mathcal{J}_C \ni [f] \ \mapsto \ \iota^* f \in \pi^* \mathcal{C}^\infty(M_{\mathrm{red}}) = \mathcal{A}_{red} \tag{2}$$

induces an isomorphism of Poisson algebras. We recall in Section 2.2 this isomorphism in our setting of Marsden Weinstein reduction using the Koszul complex.

Passing to a deformation quantized version of phase space reduction, one starts with a formal star product $\star$ on M. The associative algebra $\boldsymbol{\mathcal{A}} = (\mathcal{C}^\infty(M)[[\lambda]], \star)$ is playing the role of the quantized observables of the big system. A good analog of the vanishing ideal $\mathcal{J}_C$ will be a left ideal $\boldsymbol{\mathcal{J}}_C \subseteq \mathcal{C}^\infty(M)[[\lambda]]$ such that the quotient $\mathcal{C}^\infty(M)[[\lambda]]/\boldsymbol{\mathcal{J}}_C$ is in $\mathbb{C}[[\lambda]]$-linear bijection to the functions $\mathcal{C}^\infty(C)[[\lambda]]$ on C. Then we define $\boldsymbol{\mathcal{B}}_C = \{a \in \boldsymbol{\mathcal{A}} \mid [a, \boldsymbol{\mathcal{J}}_C] \subseteq \boldsymbol{\mathcal{J}}_C\}$, i.e. the normalizer of $\boldsymbol{\mathcal{J}}_C$ with respect to the commutator Lie bracket of $\boldsymbol{\mathcal{A}}$, and consider the associative algebra $\boldsymbol{\mathcal{B}}_C/\boldsymbol{\mathcal{J}}_C$ as the reduced algebra $\boldsymbol{\mathcal{A}}_{red}$. Of course, this is only meaningful if one can show that $\boldsymbol{\mathcal{B}}_C/\boldsymbol{\mathcal{J}}_C$ is in $\mathbb{C}[[\lambda]]$-linear bijection to $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ in such a way, that the isomorphism induces a star product $\star_{\mathrm{red}}$ on M_{red}. Starting from a strongly invariant star product on M, we describe in Section 3.1 a method to construct a good left ideal inspired by the BRST approach in [6] but simpler as we only need the deformation of the Koszul part of the BRST complex.

The algebra of quantum observables is not only an associative algebra but is has a *-involution; in the usual picture, where observables are represented by operators, this *-involution corresponds to the passage to the adjoint operator. In the framework of deformation quantization, a way to have a *-involution on $\boldsymbol{\mathcal{A}} = (C^\infty(M)[[\lambda]], \star)$ is to ask the star product to be Hermitian, i.e such that $\overline{f \star g} = \overline{g} \star \overline{f}$ and the *-involution is then just given by complex conjugation. A first question that we discuss in this paper is how to get in a natural way a *-involution for the reduced algebra, assuming that $\star$ is a Hermitian star product on M. We want a construction coming from the reduction process itself; we start with a left ideal $\boldsymbol{\mathcal{J}} \subseteq \boldsymbol{\mathcal{A}}$ in some algebra and take $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}$ as the reduced algebra, where $\boldsymbol{\mathcal{B}}$ is the normaliser of $\boldsymbol{\mathcal{J}}$ in $\boldsymbol{\mathcal{A}}$. If now $\boldsymbol{\mathcal{A}}$ is in addition a *-algebra we have to construct a *-involution for $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}$. From all relevant examples in deformation quantization one knows that $\boldsymbol{\mathcal{J}}$ is only a left ideal, hence can not be a *-ideal and thus $\boldsymbol{\mathcal{B}}$ can not be a *-subalgebra. Consequently, there is no obvious way to define a *-involution on the quotient.

The main idea here is to use a representation of the reduced quantum algebra and to translate the notion of the adjoint. Observe that $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}$ can be identified (with the opposite algenbra structure) to the algebra of $\boldsymbol{\mathcal{A}}$-linear endomorphisms of $\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{J}}$. We shall use an additional positive linear functional i.e. a $\mathbb{C}[[\lambda]]$-linear functional $\omega : \boldsymbol{\mathcal{A}} \longrightarrow \mathbb{C}[[\lambda]]$ such that $\omega(a^*a) \geq 0$ for all $a \in \boldsymbol{\mathcal{A}}$, where positivity in $\mathbb{C}[[\lambda]]$ is defined using the canonical ring ordering of $\mathbb{R}[[\lambda]]$. Defining the Gel'fand ideal of ω by $\boldsymbol{\mathcal{J}}_\omega = \left\{ a \in \boldsymbol{\mathcal{A}} \;\middle|\; \omega(a^*a) = 0 \right\}$, one can construct a *-representation (the GNS representation), of $\boldsymbol{\mathcal{A}}$ on $\mathcal{H}_\omega = \boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{J}}_\omega$ with the pre Hilbert space structure defined via $\langle \psi_a, \psi_b \rangle = \omega(a^*b)$ where ψ_a denotes the equivalence class of $a \in \boldsymbol{\mathcal{A}}$. Then the algebra of $\boldsymbol{\mathcal{A}}$-linear endomorphisms of $\mathcal{H}_\omega$ (with the opposite structure) is equal to $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}_\omega$. Hence, to define a *-involution on our reduced quantum algebra, the main idea is now to look for a positive linear functional ω such that the left ideal $\boldsymbol{\mathcal{J}}$ we use for reduction coincides with the Gel'fand ideal $\boldsymbol{\mathcal{J}}_\omega$ and such that all left $\boldsymbol{\mathcal{A}}$-linear endomorphisms of $\mathcal{H}_\omega$ are adjointable. In this case $\boldsymbol{\mathcal{B}}/\boldsymbol{\mathcal{J}}$ becomes in a natural way a *-subalgebra of the set $\mathfrak{B}(\mathcal{H}_\omega)$ of adjointable maps. Up to here, the construction is entirely algebraic and works for *-algebras over rings of the form $\mathsf{C} = \mathsf{R}(\mathrm{i})$ with $\mathrm{i}^2 = -1$ and an ordered ring R, instead of $\mathbb{C}[[\lambda]]$ and $\mathbb{R}[[\lambda]]$.

We show in Section 4.3 that the choice of a formal series of smooth densities $\sum_{r=0}^{\infty} \lambda^r \mu_r \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*C)[[\lambda]]$ on the coisotropic submanifold C such that $\overline{\mu} = \mu$ is real, $\mu_0 > 0$ and so that μ transforms under the G-action as $\mathsf{L}^*_{g^{-1}}\mu = \frac{1}{\Delta(g)}\mu$ where Δ is the modular function yields a positive linear functional which defines a *-involution on the reduced space. Along the way we identify the corresponding GNS representation. We show that in the classical Marsden Weinstein reduction, complex conjugation is a *-involution of the reduced quantum algebra. Looking in general to the question whether the *-involution corresponding to a series of densities μ is the complex conjugation yields a new notion of quantized unimodular class.

The next problem that we tackle in this paper is the study of the representations of the reduced algebra with the *-involution given by complex conjugation. We want to relate the categories of modules of the big algebra and the reduced algebra. The usual idea is to use a bimodule and the tensor product to pass from modules of one algebra to modules of the other. In the context of quantization and reduction this point of view has been pushed forward by Landsman [21], mainly in the context of geometric quantization. Contrary to his approach, we have, by construction of the reduced star product, a bimodule structure on $\mathcal{C}^\infty(C)[[\lambda]]$. We want more properties to have a relation between the *-representations of our algebras on inner product modules. The notions are transferred, following [9, 11], from the theory of Hilbert modules over C^*-algebras to our more algebraic framework and are recalled in Sections 5.1 and 6.1.

We look at $\mathcal{C}^\infty_{\mathrm{cf}}(C) = \left\{ \phi \in \mathcal{C}^\infty(C) \;\middle|\; \operatorname{supp}(\phi) \cap \pi^{-1}(K) \text{ is compact for all compact } K \subseteq M_{\mathrm{red}} \right\}$; then $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ is a left $(\mathcal{C}^\infty(M)[[\lambda]], \star)$-module and a right $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$-module; we define on it a $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued inner product and we establish in Section 6 that this bimodule structure and inner product on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ gives a strong Morita equivalence bimodule between $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ and the finite rank operators on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$. The crucial point is here to show the complete positivity of the inner product. In some sense, the resulting equivalence bimodule can be viewed as a deformation of the corresponding classical limit which is studied independently in the context of the strong Morita equivalence of the crossed product algebra with the reduced algebra. If G is not finite, the finite rank operators do not have a unit, thus we have a first non-trivial example of a *-equivalence bimodule for star product algebras going beyond the unital case studied in [10].

We show that the *-algebra $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ acts on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ in an adjointable way with respect to the $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued inner product and we obtain a Rieffel induction functor from the strongly non-degenerate *-representations of $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$ on pre-Hilbert right $\mathcal{D}$-modules to those of $(\mathcal{C}^\infty(M)[[\lambda]], \star)$, for any auxiliary coefficient *-algebra $\mathcal{D}$ over $\mathbb{C}[[\lambda]]$.

In Section 7, we consider the geometrically trivial situation $M = M_{\mathrm{red}} \times T^*G$ where on M_{red} a Poisson bracket and a corresponding star product $\star_{\mathrm{red}}$ is given while on T^*G we use the canonical symplectic Poisson structure and the canonical star product $\star_{\mathrm{G}}$ from [18]. Up to the completion issues, the Rieffel induction with $\mathcal{C}^\infty_{\mathrm{cf}}(M_{\mathrm{red}} \times G)[[\lambda]]$ simply consists in tensoring the given *-representation of $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ with the Schrödinger representation (see (144)) on $\mathcal{C}^\infty_0(G)[[\lambda]]$.

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2 The classical construction

In this section we recall some basic features of phase space reduction in order to establish our notation. The material is entirely standard, we essentially follow [6].

2.1 The geometric framework

Throughout this paper, M will denote a Poisson manifold with Poisson bracket $\{\cdot, \cdot\}$ coming from a real Poisson tensor. Thus the complex-valued functions $\mathcal{C}^\infty(M)$ on M become a Poisson *-algebra with respect to $\{\cdot, \cdot\}$ and the pointwise complex conjugation $f \mapsto \overline{f}$ as *-involution.

Let $\iota : C \hookrightarrow M$ be an embedded submanifold and denote by $\mathcal{J}_C = \{f \in \mathcal{C}^\infty(M) \mid \iota^* f = 0\} = \ker \iota^*$ the vanishing ideal of C which is an ideal in the associative algebra $\mathcal{C}^\infty(M)$. Then C is called *coisotropic* (or *first class constraint*) if $\mathcal{J}_C$ is a Poisson subalgebra of $\mathcal{C}^\infty(M)$. In this case we define

$$\mathcal{B}_C = \left\{ f \in \mathcal{C}^\infty(M) \;\middle|\; \{f, \mathcal{J}_C\} \subseteq \mathcal{J}_C \right\}, \tag{3}$$

which turns out to be the largest Poisson subalgebra of $\mathcal{C}^\infty(M)$ which contains $\mathcal{J}_C$ as a Poisson ideal. The geometric meaning of $\mathcal{B}_C$ is now the following: since C is coisotropic we have a canonical foliation of C which we assume to have a nice leaf space M_{red}. More technically, we assume that M_{red} can be equipped with the structure of a smooth manifold such that the canonical projection

$$\pi : C \longrightarrow M_{\mathrm{red}} \tag{4}$$

is a submersion. In this case one knows that M_{red} is a Poisson manifold in a canonical way such that

$$\mathcal{B}_C / \mathcal{J}_C \ni [f] \mapsto \iota^* f \in \pi^* \mathcal{C}^\infty(M_{\mathrm{red}}) \tag{5}$$

induces an isomorphism of Poisson algebras, see e.g. [5, 6]. In fact, we will give a detailed proof of this in some more particular situation later.

While in principle, phase space reduction and its deformation quantization analogs are interesting for general coisotropic submanifolds, we shall consider only a very particular case, the

Marsden-Weinstein reduction: let $\mathsf{L} : G \times M \longrightarrow M$ be a smooth left action of a connected Lie group G on M by Poisson diffeomorphisms. Moreover, assume we have an ad^*-equivariant momentum map

$$J : M \longrightarrow \mathfrak{g}^* \tag{6}$$

for this action, i.e. an ad^*-equivariant smooth map with values in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G such that the Hamiltonian vector field $X_{J_\xi} = \{\cdot, J_\xi\}$ for $J_\xi \in \mathcal{C}^\infty(M)$ with $J_\xi(p) = \langle J(p), \xi \rangle$ coincides with the fundamental vector field $\xi_M \in \Gamma^\infty(TM)$ for all $\xi \in \mathfrak{g}$. We use the convention that $\xi \mapsto \mathscr{L}_{\xi_M}$ defines an anti-homomorphism of Lie algebras, i.e.

$$\xi_M(p) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} \mathsf{L}_{\exp(t\xi)}(p) \tag{7}$$

for all $p \in M$. The ad^*-equivariance can be expressed by

$$\{J_\xi, J_\eta\} = J_{[\xi,\eta]} \tag{8}$$

for all $\xi, \eta \in \mathfrak{g}$ and it is equivalent to Ad^*-equivariance with respect to G as G is connected.

The constraint manifold C is now chosen to be the level surface of J for momentum $0 \in \mathfrak{g}^*$. Thus we assume that 0 is a value and, for simplicity, that 0 is even a *regular value*. Then

$$C = J^{-1}(\{0\}) \tag{9}$$

is an embedded submanifold which turns out to be coisotropic. The group G acts on C as well since 0 is Ad^*-invariant. We use the same symbol L for this action. The quotient (4) turns out to be just the orbit space of this group action of G on C, i.e.

$$\pi : C \longrightarrow M_{\mathrm{red}} = C/G. \tag{10}$$

In order to guarantee a good quotient we assume that G acts *freely* and *properly*: in this case C is a principal G-bundle over M_{red} and (10) is a surjective submersion as wanted. To be conform with the usual principal bundle literature, sometimes we pass to the corresponding right action of G on C given by $\mathsf{R} : C \times G \longrightarrow C$ with $\mathsf{R}_g(p) = \mathsf{L}_{g^{-1}}(p)$ as usual. Note however, that ξ_M as well as ξ_C are the fundamental vector fields with respect to the *left* actions on M and C, respectively, as in (7).

2.2 The classical Koszul resolution

For C we can now define the classical Koszul resolution. As a complex we consider $\mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g}) = \mathcal{C}^\infty(M) \otimes \Lambda^\bullet_{\mathbb{C}}\mathfrak{g}$ with the canonical free $\mathcal{C}^\infty(M)$-module structure. The group G acts on $\mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})$ by the combined action of G on the manifold and the adjoint action on $\mathfrak{g}$ extended to $\Lambda^\bullet_{\mathbb{C}}\mathfrak{g}$ by automorphisms of the $\wedge$-product. We shall denote this G-action and the corresponding $\mathfrak{g}$-action by ϱ. The *Koszul differential* is now defined by

$$\partial x = \mathrm{i}(J)x, \tag{11}$$

where $x \in \mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})$ and $\mathrm{i}(J)$ denotes the insertion of J at the first position in the $\Lambda^\bullet_{\mathbb{C}}\mathfrak{g}$-part of x. If $e_1, \ldots, e_N \in \mathfrak{g}$ denotes a basis with dual basis $e^1, \ldots, e^N \in \mathfrak{g}^*$ then we can write $J = J_a e^a$ with scalar functions $J_a \in \mathcal{C}^\infty(M)$. Here and in the following we shall use Einstein's summation convention. The Koszul differential is then

$$\partial x = J_a\, \mathrm{i}(e^a)x. \tag{12}$$

Clearly, ∂ is a super derivation of the canonical $\wedge$-product on $\mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})$ of degree -1 and $\partial^2 = 0$. Moreover, ∂ is $\mathcal{C}^\infty(M)$-linear hence we have a complex of free $\mathcal{C}^\infty(M)$-modules. Sometimes we write ∂_k for the restriction of ∂ to the antisymmetric degree $k \geq 1$.

Before proving that this indeed gives an acyclic complex we make some further simplifying assumptions needed later in the quantum version. We assume that G acts properly not only on C but on all of M. In this case we can find an open neighbourhood $M_{\mathrm{nice}} \subseteq M$ of C with the following properties: there exists a G-equivariant diffeomorphism

$$\Phi : M_{\mathrm{nice}} \longrightarrow U_{\mathrm{nice}} \subseteq C \times \mathfrak{g}^* \tag{13}$$

onto an open neighbourhood U_{nice} of $C \times \{0\}$, where the G-action on $C \times \mathfrak{g}^*$ is the product action of the one on C and Ad^*, such that for each $p \in C$ the subset $U_{\mathrm{nice}} \cap (\{p\} \times \mathfrak{g}^*)$ is star-shaped around the origin $\{p\} \times \{0\}$ and the momentum map J is given by the projection onto the second factor, i.e. $J\big|_{M_{\mathrm{nice}}} = \mathrm{pr}_2 \circ \Phi$. For a proof of this well-known fact see e.g. [6, Lem. 3].

We can use this particular tubular neighbourhood M_{nice} of C to define the following *prolongation map*

$$\mathrm{prol} : \mathcal{C}^\infty(C) \ni \phi \mapsto \mathrm{prol}(\phi) = (\mathrm{pr}_1 \circ \Phi)^* \phi \in \mathcal{C}^\infty(M_{\mathrm{nice}}). \tag{14}$$

By the equivariance of the diffeomorphism Φ the prolongation is G-equivariant as well, i.e. for $g \in G$ we have

$$\mathsf{L}_g^* \,\mathrm{prol}(\phi) = \mathrm{prol}(\mathsf{L}_g^* \phi). \tag{15}$$

The prolongation deserves its name as clearly we have for all $\phi \in \mathcal{C}^\infty(C)$

$$\iota^* \,\mathrm{prol}(\phi) = \phi. \tag{16}$$

The last ingredient from the classical side is the following homotopy which we also define only on M_{nice} for convenience. Let $x \in \mathcal{C}^\infty(M_{\mathrm{nice}}, \Lambda^k_{\mathbb{C}}\mathfrak{g})$. Since U_{nice} is star-shaped, we set

$$(h_k x)(p) = e_a \wedge \int_0^1 t^k \frac{\partial (x \circ \Phi^{-1})}{\partial \mu_a}(c, t\mu) \,\mathrm{d}\, t, \tag{17}$$

where $\Phi(p) = (c, \mu)$ for $p \in M_{\mathrm{nice}}$ and μ_a denote the linear coordinates on $\mathfrak{g}^*$ with respect to the basis $e^1, \ldots, e^N$. The collection of all these maps h_k gives a map

$$h : \mathcal{C}^\infty(M_{\mathrm{nice}}, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g}) \longrightarrow \mathcal{C}^\infty(M_{\mathrm{nice}}, \Lambda^{\bullet+1}_{\mathbb{C}}\mathfrak{g}), \tag{18}$$

whose properties are summarized in the following proposition, see e.g. [6, Lem. 5 & 6]:

Proposition 2.1 *The Koszul complex $(\mathcal{C}^\infty(M_{\mathrm{nice}}, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g}), \partial)$ is acyclic with explicit homotopy h and homology $\mathcal{C}^\infty(C)$ in degree 0. In detail, we have*

$$h_{k-1}\partial_k + \partial_{k+1} h_k = \mathsf{id}_{\mathcal{C}^\infty(M_{\mathrm{nice}}, \Lambda^k_{\mathbb{C}}\mathfrak{g})} \tag{19}$$

for $k \geq 1$ and

$$\mathrm{prol}\,\iota^* + \partial_1 h_0 = \mathsf{id}_{\mathcal{C}^\infty(M_{\mathrm{nice}})} \tag{20}$$

as well as $\iota^*\partial_1 = 0$. Thus the Koszul complex is a free resolution of $\mathcal{C}^\infty(C)$ as $\mathcal{C}^\infty(M_{\mathrm{nice}})$-module. We have

$$h_0 \,\mathrm{prol} = 0, \tag{21}$$

and all the homotopies h_k are G-equivariant.

Here resolution means that the homology at $k = 0$ is isomorphic to $\mathcal{C}^\infty(C)$ as a $\mathcal{C}^\infty(M_{\mathrm{nice}})$-module: indeed, the image of ∂_1 is just $\mathcal{J}_C \cap \mathcal{C}^\infty(M_{\mathrm{nice}})$ as (20) shows. This gives immediately

$$\mathcal{C}^\infty(M_{\mathrm{nice}}) \big/ (\mathcal{J}_C \cap \mathcal{C}^\infty(M_{\mathrm{nice}})) = \ker \partial_0 \big/ (\mathcal{J}_C \cap \mathcal{C}^\infty(M_{\mathrm{nice}})) \cong \mathcal{C}^\infty(C), \tag{22}$$

induced via ι^* and prol.

It will be useful to consider the augmented Koszul complex where in degree $k = -1$ one puts $\mathcal{C}^\infty(C)$ and re-defines $\partial_0 = \iota^*$. With $h_{-1} = \mathrm{prol}$ the proposition yields

$$h_{k-1}\partial_k + \partial_{k+1}h_k = \mathsf{id}_k \tag{23}$$

for all $k \geq -1$. This augmented complex has now trivial homology in *all* degrees.

We can use the Koszul complex to prove (5): indeed, for $u \in \mathcal{C}^\infty(M_{\mathrm{red}})$ we have $\mathrm{prol}(\pi^* u) \in \mathcal{B}_C$ whence (5) is surjective. The injectivity of (5) is clear by definition. The Poisson bracket on M_{red} can then be defined through (5) and gives explicitly

$$\pi^*\{u, v\}_{\mathrm{red}} = \iota^*\{\mathrm{prol}(\pi^* u), \mathrm{prol}(\pi^* v)\} \tag{24}$$

for $u, v \in \mathcal{C}^\infty(M_{\mathrm{red}})$, since the left hand side of (5) is canonically a Poisson algebra.

Remark 2.2 (M versus M_{nice}) For simplicity, we have defined prol as well as the homotopy h only on the neighbourhood M_{nice}. In [6] it was shown that one can extend the definitions to all of M preserving the G-equivariance and the properties (19), (20), and (21). Since for the phase space reduction in deformation quantization we will only need a very small neighbourhood (in fact: an infinitesimal one) of C, the neighbourhood M_{nice} is completely sufficient. The geometry of M far away from C will play no role in the following. Thus we may even assume $M_{\mathrm{nice}} = M$ without restriction in the following to simplify our notation.

3 The quantized bimodule structure

When passing to a deformation quantized version of phase space reduction we have to reformulate everything in terms of now non-commutative algebras where Poisson brackets are to be replaced by commutators. We recall here a general approach to reduction as proposed by Bordemann [5] as well as by Cattaneo and Felder [12, 13] and others, see also [22].

Thus in the following, let $\star$ be a formal star product [4] on M, i.e. a $\mathbb{C}[[\lambda]]$-bilinear associative multiplication for $\mathcal{C}^\infty(M)[[\lambda]]$ with

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f, g), \tag{25}$$

where $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}$. Moreover, we assume that $\star$ is bidifferential and satisfies $1 \star f = f = f \star 1$ for all $f \in \mathcal{C}^\infty(M)[[\lambda]]$. Physically speaking, the formal parameter λ corresponds to Planck's constant $\hbar$ whenever we can establish convergence of the above formal series, see e.g. [15] for a review on deformation quantization and [27] for a gentle introduction.

The first observation is that a good analog of the vanishing ideal $\mathcal{J}_C$ will be a *left ideal*: this is Dirac's old ideal of "weakly vanishing operators" annihilating the "true physical states" inside some "unphysical, too big Hilbert space", see [14] as well as [22]. Thus the general situation is to have an associative algebra $\mathcal{A}$ playing the role of the observables of the big system with a left ideal $\mathcal{J} \subseteq \mathcal{A}$. The functions on the constraint surface will correspond to the left $\mathcal{A}$-module $\mathcal{A}/\mathcal{J}$ in the non-commutative world. The following simple proposition gives now a nice motivation how to define the reduced algebra, i.e. the observables of the reduced system:

Proposition 3.1 *Let $\mathcal{A}$ be a unital algebra with a left ideal $\mathcal{J} \subseteq \mathcal{A}$. Define*

$$\mathcal{B} = \{a \in \mathcal{A} \mid [a, \mathcal{J}] \subseteq \mathcal{J}\}, \tag{26}$$

i.e. the normalizer of $\mathcal{J}$ with respect to the commutator Lie bracket of $\mathcal{A}$. Then $\mathcal{B}$ is the largest unital subalgebra of $\mathcal{A}$ such that $\mathcal{J} \subseteq \mathcal{B}$ is a two-sided ideal and

$$\mathcal{B}/\mathcal{J} \ni [b] \mapsto ([a] \mapsto [ab]) \in \mathsf{End}_{\mathcal{A}}\left(\mathcal{A}/\mathcal{J}\right)^{\mathrm{opp}} \tag{27}$$

is an isomorphism of unital algebras.

This observation gives now the guideline for the reduction of star products: for the star product $\star$ on M, we have to find a left ideal $\mathcal{J}_C \subseteq \mathcal{C}^\infty(M)[[\lambda]]$ such that the quotient $\mathcal{C}^\infty(M)[[\lambda]]/\mathcal{J}_C$ is in $\mathbb{C}[[\lambda]]$-linear bijection to the functions $\mathcal{C}^\infty(C)[[\lambda]]$ on C. Then we consider the associative algebra $\mathcal{B}_C/\mathcal{J}_C$ as the reduced algebra. Of course, this is only meaningful if one can show that $\mathcal{B}_C/\mathcal{J}_C$ is in $\mathbb{C}[[\lambda]]$-linear bijection to $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ in such a way, that the isomorphism induces a star product $\star_{\mathrm{red}}$ on M_{red}. This is the general reduction philosophy as proposed by [5, 6, 12, 13, 22] which makes sense for general coisotropic submanifolds. We note that as a result one obtains even a *bimodule* structure on $\mathcal{C}^\infty(C)[[\lambda]]$ where $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ acts from the left and $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$ acts from the right. Note also that the situation will be quite asymmetric in general: while all left $\star$-linear endomorphisms are indeed given by right multiplications with functions in $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ according to Proposition 3.1, the converse needs not to be true in general: In fact, we will see explicit counter-examples later.

3.1 The quantized Koszul complex

We describe now a method how to construct a left ideal and a deformed left module structure for the functions on C inspired by the BRST approach in [6]. However, for us things will be slightly simpler as we only need the Koszul part of the BRST complex.

Before defining the deformed Koszul operator we have to make some further assumptions on the star product $\star$ on M. First, we want it to be $\mathfrak{g}$-*covariant*, i.e.

$$J_\xi \star J_\eta - J_\eta \star J_\xi = \mathrm{i}\lambda J_{[\xi,\eta]} \tag{28}$$

for all $\xi, \eta \in \mathfrak{g}$. Second, we need $\star$ to be G-*invariant*, i.e.

$$\mathsf{L}_g^*(f \star h) = (\mathsf{L}_g^* f) \star (\mathsf{L}_g^* h) \tag{29}$$

for all $g \in G$ and $f, h \in \mathcal{C}^\infty(M)[[\lambda]]$. In general, both conditions are quite independent but there is one way to guarantee both features: we ask for a *strongly invariant* star product, see also [2]. This means

$$J_\xi \star f - f \star J_\xi = \mathrm{i}\lambda \{J_\xi, f\} = -\mathrm{i}\lambda \mathscr{L}_{\xi_M} f \tag{30}$$

for all $f \in \mathcal{C}^\infty(M)[[\lambda]]$ and $\xi \in \mathfrak{g}$. Indeed, (30) clearly implies (28) by taking $f = J_\eta$ using (8). Since the left hand side of (30) is a (quasi-inner) derivation of $\star$ so is the right hand side. Thus the invariance (29) follows by differentiation of $g = \exp(t\xi)$ as usual. Note that G is assumed to be connected in the context of phase space reduction.

Using the $\wedge$-product for $\Lambda^\bullet_{\mathbb{C}}\mathfrak{g}$ we extend $\star$ to $\mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})$ in the canonical way. This allows for the following definition:

Definition 3.2 (Quantized Koszul operator) *Let $\kappa \in \mathbb{C}[[\lambda]]$. The quantized Koszul operator $\boldsymbol{\partial}^{(\kappa)} : \mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})[[\lambda]] \longrightarrow \mathcal{C}^\infty(M, \Lambda^{\bullet+1}_{\mathbb{C}}\mathfrak{g})[[\lambda]]$ is defined by*

$$\boldsymbol{\partial}^{(\kappa)} x = \mathrm{i}(e^a) x \star J_a + \frac{\mathrm{i}\lambda}{2} C_{ab}^c e_c \wedge \mathrm{i}(e^a)\, \mathrm{i}(e^b) x + \mathrm{i}\lambda\kappa\, \mathrm{i}(\Delta) x, \tag{31}$$

where $C_{ab}^c = e^c([e_a, e_b])$ are the structure constants of $\mathfrak{g}$ and

$$\Delta(\xi) = \operatorname{tr} \operatorname{ad}(\xi) \quad \textit{for} \quad \xi \in \mathfrak{g} \tag{32}$$

is the modular one-form $\Delta \in \mathfrak{g}^*$ of $\mathfrak{g}$.

Note that with respect to the chosen basis we have

$$\Delta = C_{ab}^b e^a. \tag{33}$$

Lemma 3.3 *Let $\star$ be strongly invariant and $\kappa \in \mathbb{C}[[\lambda]]$.*

i.) One has $\boldsymbol{\partial}^{(0)}\,\mathrm{i}(\Delta) + \mathrm{i}(\Delta)\boldsymbol{\partial}^{(0)} = 0$.

ii.) $\boldsymbol{\partial}^{(\kappa)}$ is left $\star$-linear.

iii.) The classical limit of $\boldsymbol{\partial}^{(\kappa)}$ is ∂.

iv.) $\boldsymbol{\partial}^{(\kappa)}$ is G-equivariant.

v.) $\boldsymbol{\partial}^{(\kappa)} \circ \boldsymbol{\partial}^{(\kappa)} = 0$.

Proof. For the first part we note that the insertion of the *constant* one-form $\Delta \in \mathfrak{g}^*$ anti-commutes with the first part of $\boldsymbol{\partial}^{(0)}$. It also anti-commutes with the second part as Δ vanishes on Lie brackets. The second and third part is clear. The fourth part is a simple computation. For the last part it is sufficient to consider the case $\kappa = 0$ which is a straightforward computation using the covariance of $\star$. Then $\mathrm{i}(\Delta)\,\mathrm{i}(\Delta) = 0$ and the first part give also the general case $\kappa \in \mathbb{C}[[\kappa]]$. □

The importance of the correction term $\mathrm{i}\lambda\kappa\,\mathrm{i}(\Delta)$ will become clear in Section 4.2. For the time being, κ can be arbitrary. In particular, $\kappa = 0$ gives a very simple choice for the quantized Koszul operator. However, we set

$$\boldsymbol{\partial} = \boldsymbol{\partial}^{(\kappa=\frac{1}{2})} \tag{34}$$

for abbreviation as this value of κ will turn out to be the most useful choice. The following constructions will always depend on κ. If we omit the reference to κ in our notation, we always mean the particular value of κ as in (34).

Following [6] we obtain a deformation of the restriction map ι^* as follows. We define

$$\boldsymbol{\iota}_\kappa^* = \iota^* \left(\mathsf{id} + \left(\boldsymbol{\partial}_1^{(\kappa)} - \partial_1\right) h_0\right)^{-1} : \mathcal{C}^\infty(M)[[\lambda]] \longrightarrow \mathcal{C}^\infty(C)[[\lambda]] \tag{35}$$

and

$$\boldsymbol{h}_0^{(\kappa)} = h_0 \left(\mathsf{id} + \left(\boldsymbol{\partial}_1^{(\kappa)} - \partial_1\right) h_0\right)^{-1} : \mathcal{C}^\infty(M)[[\lambda]] \longrightarrow \mathcal{C}^\infty(M, \mathfrak{g})[[\lambda]], \tag{36}$$

which are both well-defined since $\boldsymbol{\partial}^{(\kappa)}$ is a deformation of ∂. From [6, Prop. 25] we know that

$$\boldsymbol{h}_0^{(\kappa)}\,\mathrm{prol} = 0, \quad \boldsymbol{\iota}_\kappa^* \boldsymbol{\partial}_1^{(\kappa)} = 0, \quad \text{and} \quad \boldsymbol{\iota}_\kappa^*\,\mathrm{prol} = \mathsf{id}_{\mathcal{C}^\infty(C)[[\lambda]]}\,. \tag{37}$$

Analogously to the definition of $\boldsymbol{h}_0^{(\kappa)}$ one can also deform the higher homotopies h_k by setting

$$\boldsymbol{h}_k^{(\kappa)} = h_k \left(h_{k-1}\boldsymbol{\partial}_k^{(\kappa)} + \boldsymbol{\partial}_{k+1}^{(\kappa)} h_k\right)^{-1}, \tag{38}$$

for which one obtains the following properties [6]:

Lemma 3.4 *The deformed augmented Koszul complex, where $\boldsymbol{\partial}_0^{(\kappa)} = \boldsymbol{\iota}_\kappa^*$, has trivial homology: with $\boldsymbol{h}_{-1}^{(\kappa)} = \mathrm{prol}$ one has*

$$\boldsymbol{h}_{k-1}^{(\kappa)} \boldsymbol{\partial}_k^{(\kappa)} + \boldsymbol{\partial}_{k+1}^{(\kappa)} \boldsymbol{h}_k^{(\kappa)} = \mathsf{id}_{\mathcal{C}^\infty(M, \Lambda_{\mathbb{C}}^k \mathfrak{g})[[\lambda]]} \tag{39}$$

for $k \geq 0$ and $\boldsymbol{\iota}_\kappa^*\,\mathrm{prol} = \mathsf{id}_{\mathcal{C}^\infty(C)[[\lambda]]}$ for $k = -1$. Moreover, the maps $\boldsymbol{\iota}_\kappa^*$ and $\boldsymbol{h}_k^{(\kappa)}$ are G-equivariant.

For $k = 0$ the homotopy equation (38) becomes explicitly

$$\mathrm{prol}\,\boldsymbol{\iota}_\kappa^* + \boldsymbol{\partial}_1^{(\kappa)} \boldsymbol{h}_0^{(\kappa)} = \mathsf{id}_{\mathcal{C}^\infty(M)[[\lambda]]}\,. \tag{40}$$

In fact, we will only need this part of the Koszul resolution. Finally, we mention the following locality feature of $\boldsymbol{\iota}_\kappa^*$ which is remarkable since the homotopy h_0 used in (35) is *not* local, see [6, Lem. 27]:

Lemma 3.5 *There is a formal series $S_\kappa = \mathsf{id} + \sum_{r=1}^\infty \lambda^r S_r^{(\kappa)}$ of G-invariant differential operators $S_r^{(\kappa)}$ on M such that*

$$\boldsymbol{\iota}_\kappa^* = \iota^* \circ S_\kappa. \tag{41}$$

Moreover, S_κ can be arranged such that $S_\kappa 1 = 1$.

3.2 The reduced star product and the bimodule

Let us now use the deformed homotopy equation (39) to construct the bimodule structure on $\mathcal{C}^\infty(C)[[\lambda]]$. This construction is implicitly available in [6], see also [5] for a more profound discussion.

Definition 3.6 *The deformed left multiplication of $\phi \in \mathcal{C}^\infty(C)[[\lambda]]$ by some $f \in \mathcal{C}^\infty(M)[[\lambda]]$ is defined by*

$$f \bullet_\kappa \phi = \boldsymbol{\iota}_\kappa^*(f \star \mathrm{prol}(\phi)). \tag{42}$$

This defines a left module structure indeed. Moreover, it has nice locality and invariance properties which we summarize in the following proposition:

Proposition 3.7 *Let $\mathfrak{J}_C = \mathrm{im}\, \boldsymbol{\partial}_1^{(\kappa)}$ be the image of the Koszul differential.*

i.) $\mathfrak{J}_C$ is a left $\star$-ideal.

ii.) The left module $\mathcal{C}^\infty(M)[[\lambda]]/\mathfrak{J}_C$ is isomorphic to $\mathcal{C}^\infty(C)[[\lambda]]$ equipped with $\bullet_\kappa$ via the mutually inverse isomorphisms

$$\mathcal{C}^\infty(M)[[\lambda]]/\mathfrak{J}_C \ni [f] \mapsto \boldsymbol{\iota}_\kappa^* f \in \mathcal{C}^\infty(C)[[\lambda]] \tag{43}$$

and

$$\mathcal{C}^\infty(C)[[\lambda]] \ni \phi \mapsto [\mathrm{prol}(\phi)] \in \mathcal{C}^\infty(M)[[\lambda]]/\mathfrak{J}_C. \tag{44}$$

iii.) The left module structure $\bullet_\kappa$ is bidifferential along ι^*, i.e. we have $\mathbb{C}$-bilinear operators $L_r^{(\kappa)} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(C) \longrightarrow \mathcal{C}^\infty(C)$ with

$$f \bullet_\kappa \phi = \iota^*(f)\phi + \sum_{r=1}^{\infty} \lambda^r L_r^{(\kappa)}(f, \phi), \tag{45}$$

where $L_r^{(\kappa)}$ is differential along ι^* in the first and differential in the second argument.

iv.) The left module structure is G-invariant in the sense that

$$\mathsf{L}_g^*(f \bullet_\kappa \phi) = (\mathsf{L}_g^* f) \bullet_\kappa (\mathsf{L}_g^* \phi) \tag{46}$$

for all $g \in G$, $f \in \mathcal{C}^\infty(M)[[\lambda]]$, and $\phi \in \mathcal{C}^\infty(C)[[\lambda]]$. Moreover, we have for all $\xi \in \mathfrak{g}$

$$J_\xi \bullet_\kappa \phi = -\mathrm{i}\lambda \mathscr{L}_{\xi_C} \phi - \mathrm{i}\lambda\kappa\Delta(\xi)\phi. \tag{47}$$

Proof. For the reader's convenience we sketch the proof, see also [5, 6]. Recall that we assume $M = M_{\mathrm{nice}}$. Since $\boldsymbol{\partial}_1^{(\kappa)}$ is left $\star$-linear its image is a left ideal. Then the well-definedness of (43) and (44) follows from (40) and (37). It is clear that they are mutually inverse to each other. The canonical module structure of the left hand side of (43) transported to $\mathcal{C}^\infty(C)[[\lambda]]$ via (43) and (44) gives (42). This shows the second part. The third follows from Lemma 3.5 since $\star$ is bidifferential, too. The G-invariance is clear as $\star$, prol, and $\boldsymbol{\iota}_\kappa^*$ are G-invariant. The last part is a straightforward computation using the strong invariance (30). We have

$$\begin{aligned} J_\xi \bullet_\kappa \phi &= \boldsymbol{\iota}_\kappa^* \left(J_\xi \star \mathrm{prol}(\phi) - \mathrm{prol}(\phi) \star J_\xi + \mathrm{prol}(\phi) \star J_\xi\right) \\ &= \boldsymbol{\iota}_\kappa^* \Big(\mathrm{i}\lambda\{J_\xi, \mathrm{prol}(\phi)\} + \boldsymbol{\partial}_1^{(\kappa)} (\mathrm{prol}(\phi) \otimes \xi) - \mathrm{i}\lambda\kappa\, \mathrm{i}(\Delta)\, (\mathrm{prol}(\phi) \otimes \xi)\Big) \\ &= -\mathrm{i}\lambda \boldsymbol{\iota}_\kappa^* \mathscr{L}_{\xi_M} \mathrm{prol}(\phi) - \mathrm{i}\lambda\kappa \boldsymbol{\iota}_\kappa^* \left(\Delta(\xi)\, \mathrm{prol}(\phi)\right) = -\mathrm{i}\lambda \mathscr{L}_{\xi_C} \phi - \mathrm{i}\lambda\kappa\Delta(\xi)\phi, \end{aligned}$$

using the invariance of $\boldsymbol{\iota}_\kappa^*$ and $\boldsymbol{\iota}_\kappa^* \mathrm{prol}(\phi) = \phi$. □

Remark 3.8 Thanks to the locality features of $\boldsymbol{\iota}_\kappa^*$ and $\bullet_\kappa$ we see that only $M_{\mathrm{nice}} \subseteq M$ enters the game. Thus this justifies our previous simplification in Remark 2.2 to consider M_{nice} only and assume $M_{\mathrm{nice}} = M$ from the beginning.

Remark 3.9 Since $\boldsymbol{\partial}_\kappa$ is left $\star$-linear, it follows from (40) that $\boldsymbol{\iota}_\kappa^*$ is left $\star$-linear, i.e a module homomorphism. This way, the deformed Koszul complex becomes indeed a (free) resolution of the deformed module $(\mathcal{C}^\infty(C)[[\lambda]], \bullet_\kappa)$, see also the proof of Theorem 4.14.

Remark 3.10 From (47) we see that $\kappa = 0$ would also be a preferred choice. Note that all choices of κ are compatible with the representation property

$$J_\xi \bullet_\kappa J_\eta \bullet_\kappa \phi - J_\eta \bullet_\kappa J_\xi \bullet_\kappa \phi = \mathrm{i}\lambda J_{[\xi,\eta]} \bullet_\kappa \phi \tag{48}$$

since $\Delta(\xi)$ is a constant. Of course, (48) is also clear from (28) and $\bullet_\kappa$ being a left module structure.

From our general considerations in Proposition 3.1 we know already how to compute the module endomorphisms of the deformed module $(\mathcal{C}^\infty(C)[[\lambda]], \bullet_\kappa)$. The next proposition gives now an explicit description of the quotient $\mathcal{B}_C/\mathcal{J}_C$ where

$$\mathcal{B}_C = \left\{ f \in \mathcal{C}^\infty(M)[[\lambda]] \;\middle|\; [f, \mathcal{J}_C]_\star \subseteq \mathcal{J}_C \right\} \tag{49}$$

according to (26). This way, we also obtain the explicit form of the bimodule structure, see [6, Thm. 29 & 32]:

Proposition 3.11 *Let $f \in \mathcal{C}^\infty(M)[[\lambda]]$, $\phi \in \mathcal{C}^\infty(C)[[\lambda]]$, and $u, v \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$.*

i.) We have $f \in \mathcal{B}_C$ iff $\mathscr{L}_{\xi_C} \boldsymbol{\iota}_\kappa^* f = 0$ for all $\xi \in \mathfrak{g}$ iff $\boldsymbol{\iota}_\kappa^* f \in \pi^* \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$.

ii.) The quotient algebra $\mathcal{B}_C/\mathcal{J}_C$ is isomorphic to $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ via the mutually inverse maps

$$\mathcal{B}_C/\mathcal{J}_C \ni [f] \mapsto \boldsymbol{\iota}_\kappa^* f \in \pi^* \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]] \tag{50}$$

and

$$\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]] \ni u \mapsto [\mathrm{prol}(\pi^* u)] \in \mathcal{B}_C/\mathcal{J}_C. \tag{51}$$

iii.) The induced associative product $\star_{\mathrm{red}}^{(\kappa)}$ on $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ from $\mathcal{B}_C/\mathcal{J}_C$ is explicitly given by

$$\pi^*(u \star_{\mathrm{red}}^{(\kappa)} v) = \boldsymbol{\iota}_\kappa^* \left(\mathrm{prol}(\pi^* u) \star \mathrm{prol}(\pi^* v) \right). \tag{52}$$

This is a bidifferential star product quantizing the Poisson bracket (24).

iv.) The induced right $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}}^{(\kappa)})$-module structure $\bullet_{\mathrm{red}}^{(\kappa)}$ on $\mathcal{C}^\infty(C)[[\lambda]]$ from (27) is bidifferential and explicitly given by

$$\phi \bullet_{\mathrm{red}}^{(\kappa)} u = \boldsymbol{\iota}_\kappa^* \left(\mathrm{prol}(\phi) \star \mathrm{prol}(\pi^* u) \right). \tag{53}$$

v.) The right module structure is G-invariant, i.e. for $g \in G$ we have

$$\mathsf{L}_g^* (\phi \bullet_{\mathrm{red}}^{(\kappa)} u) = (\mathsf{L}_g^* \phi) \bullet_{\mathrm{red}}^{(\kappa)} u. \tag{54}$$

vi.) We have $1 \bullet_{\mathrm{red}}^{(\kappa)} u = \pi^* u$.

Proof. Again, we sketch the proof. For the first part note that $\mathcal{J}_C = \ker \boldsymbol{\iota}_\kappa^*$ according to (40). Now let $g = g^a \star J_a + \mathrm{i}\lambda\kappa C_{ba}^a g^b$ with $g^a \in \mathcal{C}^\infty(M)[[\lambda]]$ be in the image of $\boldsymbol{\partial}_1^{(\kappa)}$. For $f \in \mathcal{C}^\infty(M)[[\lambda]]$ we have by a straightforward computation

$$[f, g]_\star = \boldsymbol{\partial}_1^{(\kappa)} h + \mathrm{i}\lambda g^a \star \mathscr{L}_{(e_a)_M} f$$

with some $h \in \mathcal{C}^\infty(M, \mathfrak{g})$ using the strong invariance (30) of $\star$. Thus $[f,g]_\star$ is in $\boldsymbol{\mathcal{J}}_C$ iff $g^a \star \mathscr{L}_{(e_a)_M} f$ is in the image of $\boldsymbol{\partial}_1^{(\kappa)}$ for all g^a. This shows that $f \in \boldsymbol{\mathcal{B}}_C$ iff $\mathscr{L}_{\xi_M} f \in \operatorname{im} \boldsymbol{\partial}_1^{(\kappa)} = \ker \boldsymbol{\iota}_\kappa^*$. Since $\boldsymbol{\iota}_\kappa^*$ is G-invariant the first part follows. The second part is then clear from the first part and (52) is a straightforward translation using the isomorphisms (50) and (51). From (52) and Lemma 3.5 it follows that $\star_{\mathrm{red}}^{(\kappa)}$ is bidifferential. The first orders of $\star_{\mathrm{red}}^{(\kappa)}$ are easily computed showing that it is indeed a star product on M_{red}. The fourth part is clear from the second, the next part follows from the G-invariance of all involved maps. The last part is clear since $\operatorname{prol}(1) = 1$. □

Remark 3.12 From the general Proposition 3.1 we know that the right multiplications by functions $u \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ via $\bullet_{\mathrm{red}}^{(\kappa)}$ constitute precisely the module endomorphisms with respect to the left $\bullet_\kappa$-multiplications. The converse is not true: though the map

$$(\mathcal{C}^\infty(M)[[\lambda]], \star) \ni f \mapsto (\phi \mapsto f \bullet_\kappa \phi) \in \mathsf{End}_{(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}}^{(\kappa)})}\left(\mathcal{C}^\infty(C)[[\lambda]], \bullet_{\mathrm{red}}^{(\kappa)}\right) \tag{55}$$

is a homomorphism of algebras, it is neither injective nor surjective: By the locality (45) it is clear that $f \bullet_\kappa \phi = 0$ for all ϕ if all derivatives of f vanish on C. In particular, we have $f \bullet_\kappa \phi = 0$ for $\operatorname{supp} f \cap C = \emptyset$. Also, the map $\phi \mapsto \mathsf{L}_g^* \phi$ for $g \in G$ is in the module endomorphisms with respect to the right $\star_{\mathrm{red}}^{(\kappa)}$-module structure by (54). Being a *non-local* operation (unless g acts trivially on C) we conclude that it can not be of the form $\phi \mapsto f \bullet_\kappa \phi$.

4 *-Involutions by reduction

In this section we discuss how a *-involution for $\star_{\mathrm{red}}$ can be constructed. To this end we assume that $\star$ is a *Hermitian* star product on M, i.e. we have

$$\overline{f \star g} = \overline{g} \star \overline{f} \tag{56}$$

for all $f, g \in \mathcal{C}^\infty(M)[[\lambda]]$. The existence of such Hermitian star products is well-understood, see e.g. [24] for the symplectic case.

The question we would like to address is whether and how one can obtain a star product $\star_{\mathrm{red}}$ for which the complex conjugation or a suitable deformation is a *-involution. In principle, there is a rather cheap answer: one has to compute a certain characteristic class of $\star_{\mathrm{red}}$, apply the results of [24], and conclude that there is an equivalent star product to $\star_{\mathrm{red}}$ which is Hermitian. However, we want a construction coming from the reduction process itself and hence from M instead of the above more intrinsic argument. From a more conceptual point of view this is very much desirable as ultimately one wants to apply reduction procedures also to situations where nice differential geometry for M_{red}, and hence the results of e.g. [24], may fail due to singularities.

Now this approach makes things more tricky: according to our reduction philosophy we start with a left ideal $\mathcal{J} \subseteq \mathcal{A}$ in some algebra and take $\mathcal{B}/\mathcal{J}$ as the reduced algebra. If now $\mathcal{A}$ is in addition a *-algebra we have to construct a *-involution for $\mathcal{B}/\mathcal{J}$. From all relevant examples in deformation quantization one knows that $\mathcal{J}$ is *only* a left ideal. Thus $\mathcal{J}$ can not be a *-ideal and thus $\mathcal{B}$ can not be a *-subalgebra. Consequently, there is *no obvious* way to define a *-involution on the quotient $\mathcal{B}/\mathcal{J}$. In fact, some additional ingredients will be needed.

4.1 Algebraic preliminaries

The main idea to construct the *-involution is to use a representation of the reduced algebra as adjointable operators acting on a pre-Hilbert space over $\mathbb{C}[[\lambda]]$. Since the reduced algebra $\mathcal{B}/\mathcal{J}$ can be identified to the algebra $\mathsf{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{J})^{\mathrm{opp}}$ (i.e. with the opposite algebra structure), a first idea is to build a structure of pre-Hilbert space on $\mathcal{A}/\mathcal{J}$. To this aim, one considers an additional positive linear functional on $\mathcal{A}$. To put things into the correct algebraic framework we consider the following situation, see e.g. [27, Chap. 7] for more details and further references. Let R be an ordered ring and $\mathsf{C} = \mathsf{R}(\mathrm{i})$ its extension by a square root i of -1. The relevant examples for us are $\mathsf{R} = \mathbb{R}$ and $\mathsf{R} = \mathbb{R}[[\lambda]]$ with $\mathsf{C} = \mathbb{C}$ and $\mathsf{C} = \mathbb{C}[[\lambda]]$, respectively. Recall that a formal series in $\mathbb{R}[[\lambda]]$ is called positive if the lowest non-vanishing order is positive. Then let $\mathcal{A}$ be a *-algebra over C, i.e. an associative algebra equipped with a C-antilinear involutive anti-automorphism, the *-involution $^* : \mathcal{A} \longrightarrow \mathcal{A}$. The ring ordering allows now the following definition: a C-linear functional $\omega : \mathcal{A} \longrightarrow \mathsf{C}$ is called positive if for all $a \in \mathcal{A}$

$$\omega(a^* a) \geq 0. \tag{57}$$

In this case we have a Cauchy-Schwarz inequality $\omega(a^*b)\overline{\omega(a^*b)} \leq \omega(a^*a)\omega(b^*b)$ and the reality $\omega(a^*b) = \overline{\omega(b^*a)}$ as usual. It follows that

$$\mathcal{J}_\omega = \left\{ a \in \mathcal{A} \;\middle|\; \omega(a^*a) = 0 \right\} \tag{58}$$

is a left ideal in $\mathcal{A}$, the *Gel'fand ideal* of ω.

Recall that a *pre Hilbert space* $\mathcal{H}$ over C is a C-module equipped with a scalar product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathsf{C}$ which is C-linear in the second argument and satisfies $\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle$ as well as $\langle \phi, \phi \rangle > 0$ for $\phi \neq 0$. By $\mathfrak{B}(\mathcal{H})$ we denote the *adjointable operators* on $\mathcal{H}$, i.e. those maps $A : \mathcal{H} \longrightarrow \mathcal{H}$ for which there is a map $A^* : \mathcal{H} \longrightarrow \mathcal{H}$ with $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$ for all $\phi, \psi \in \mathcal{H}$. It follows that such a map is C-linear and $\mathfrak{B}(\mathcal{H})$ becomes a unital *-algebra over C. This allows to define a **-representation* of $\mathcal{A}$ on $\mathcal{H}$ to be a *-homomorphism $\pi : \mathcal{A} \longrightarrow \mathfrak{B}(\mathcal{H})$. For these notions and further references on *-representation theory we refer to [27, Chap. 7] as well as to [26] for the more particular case of *-algebras and O^*-algebras over $\mathbb{C}$.

Having a positive linear functional $\omega : \mathcal{A} \longrightarrow \mathsf{C}$ one constructs a *-representation $(\mathcal{H}_\omega, \pi_\omega)$, the *GNS representation*, of $\mathcal{A}$ as follows: setting $\mathcal{H}_\omega = \mathcal{A}/\mathcal{J}_\omega$ yields on one hand a left $\mathcal{A}$-module and on the other hand a pre Hilbert space via $\langle \psi_a, \psi_b \rangle = \omega(a^*b)$ where $\psi_a, \psi_b \in \mathcal{H}_\omega$ denote the equivalence classes of $a, b \in \mathcal{A}$. Then one checks immediately that the canonical left module structure, denoted by $\pi_\omega(a)\psi_b = \psi_{ab}$ in this context, is a *-representation of $\mathcal{A}$ on $\mathcal{H}_\omega$.

Back to our reduction problem, the main idea is now to look for a positive linear functional ω such that the left ideal $\mathcal{J}$ we use for reduction coincides with the Gel'fand ideal $\mathcal{J}_\omega$. In this case we have the following simple statement:

Proposition 4.1 *Assume $\omega : \mathcal{A} \longrightarrow \mathsf{C}$ is a positive linear functional with $\mathcal{J}_\omega = \mathcal{J}$ and hence $\mathcal{A}/\mathcal{J} = \mathcal{H}_\omega$. Then $\mathsf{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{J}) \cap \mathfrak{B}(\mathcal{H}_\omega)$ is a *-subalgebra of $\mathfrak{B}(\mathcal{H}_\omega)$ and a subalgebra of $\mathsf{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{J})$.*

Proof. In general, there may be left module endomorphisms which are not adjointable and adjointable endomorphisms which are not left $\mathcal{A}$-linear. The non-trivial part is to show that for $A \in \mathsf{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{J}) \cap \mathfrak{B}(\mathcal{H}_\omega)$ also A^* is left $\mathcal{A}$-linear. Thus let $a \in \mathcal{A}$ and $\phi, \psi \in \mathcal{H}_\omega = \mathcal{A}/\mathcal{J}$. Then

$$\langle A^*\pi_\omega(a)\phi, \psi \rangle = \langle \pi_\omega(a)\phi, A\psi \rangle = \langle \phi, \pi_\omega(a^*)A\psi \rangle = \langle \phi, A\pi_\omega(a^*)\psi \rangle = \langle \pi_\omega(a)A^*\phi, \psi \rangle,$$

since $\pi_\omega(a)^* = \pi_\omega(a^*)$. Since $\langle \cdot, \cdot \rangle$ is non-degenerate we conclude that $A^*\pi_\omega(a) = \pi_\omega(a)A^*$. □

Thus from $\mathcal{B}/\mathcal{J} \cong \mathsf{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{J})^{\mathrm{opp}}$ we obtain at least a subalgebra of $\mathcal{B}/\mathcal{J}$ which is a *-algebra with *-involution inherited from $\mathfrak{B}(\mathcal{H}_\omega)$.

Remark 4.2 In conclusion, we want a positive functional $\omega : \mathcal{A} \longrightarrow \mathsf{C}$ such that first $\mathcal{J} = \mathcal{J}_\omega$ and second *all* left $\mathcal{A}$-linear endomorphisms of $\mathcal{H}_\omega = \mathcal{A}/\mathcal{J}$ are adjointable. In this case $\mathcal{B}/\mathcal{J}$ becomes a *-subalgebra of $\mathfrak{B}(\mathcal{H}_\omega)$ in a natural way. Of course, up to now this is only an algebraic game as the existence of such a functional ω is by far not obvious.

4.2 The positive functional

While in the general algebraic situation not much can be said about the existence of a suitable positive functional, in our geometric context we can actually construct a fairly simple ω.

To this end we investigate the behaviour of the operators introduced in Section 2 and 3 under complex conjugation. On $\mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})$ we define the complex conjugation pointwise in M and require the elements of $\Lambda^\bullet_{\mathbb{R}}\mathfrak{g}$ to be real. Recall that our construction of $\star_{\mathrm{red}}$ uses a strongly invariant Hermitian star product $\star$ on M.

Lemma 4.3 *Let $x \in \mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})[[\lambda]]$. Then*

$$\overline{hx} = h\overline{x}, \quad \overline{\partial x} = \partial \overline{x}, \quad \text{and} \quad \overline{\iota^* x} = \iota^* \overline{x}, \tag{59}$$

and

$$\overline{\boldsymbol{\partial}^{(\kappa)} x} = \boldsymbol{\partial}^{(\kappa)} \overline{x} - \mathrm{i}\lambda \mathscr{L}_{(e_a)_M} \mathrm{i}(e^a)\overline{x} - \mathrm{i}\lambda C^c_{ab} e_c \wedge \mathrm{i}(e^a)\,\mathrm{i}(e^b)\overline{x} - \mathrm{i}\lambda(\overline{\kappa} + \kappa)\,\mathrm{i}(\Delta)\overline{x}. \tag{60}$$

Moreover, for $\phi \in \mathcal{C}^\infty(C)[[\lambda]]$ we have

$$\overline{\mathrm{prol}(\phi)} = \mathrm{prol}(\overline{\phi}). \tag{61}$$

Proof. The claims in (59) and (61) are trivial. For $\kappa = 0$ the claim (60) is a simple computation using again the strong invariance (30) as well as (56). But then the case $\kappa \in \mathbb{C}[[\lambda]]$ follows since the one-form Δ is real. □

Before we proceed we have to rewrite the result (60) in the following way. Since $\star$ is strongly invariant we have for the (left) action of $\mathfrak{g}$ on $\mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})$

$$\varrho(\xi)x = -\mathscr{L}_{\xi_M} x + \mathrm{ad}_\xi x = \frac{1}{\mathrm{i}\lambda}[J_\xi, x]_\star + \mathrm{ad}_\xi x \tag{62}$$

for all $x \in \mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}}\mathfrak{g})$. Using this, we get from (60)

$$\overline{\boldsymbol{\partial}^{(\kappa)} x} = \boldsymbol{\partial}^{(\kappa)} \overline{x} + \mathrm{i}\lambda\left(\varrho(e_a) - \mathrm{ad}_{e_a}\right)\mathrm{i}(e^a)\overline{x} - \mathrm{i}\lambda C^c_{ab} e_c \wedge \mathrm{i}(e^a)\,\mathrm{i}(e^b)\overline{x} - \mathrm{i}\lambda(\overline{\kappa} + \kappa)\,\mathrm{i}(\Delta)\overline{x}. \tag{63}$$

While the Lie derivative $\mathscr{L}_{\xi_M}$ commutes with all the insertions $\mathrm{i}(\alpha)$ of *constant* one-forms $\alpha \in \mathfrak{g}^*$ this is no longer true for $\varrho(\xi)$ and ad_ξ. In fact, by a simple computation we get

$$\mathrm{ad}_{e_a} \mathrm{i}(e^a)x - \mathrm{i}(e^a)\,\mathrm{ad}_{e_a} x = \mathrm{i}(\Delta)x = \varrho(e_a)\,\mathrm{i}(e^a)x - \mathrm{i}(e^a)\varrho(e_a)x \tag{64}$$

for all $x \in \mathcal{C}^\infty(M, \Lambda^\bullet_{\mathbb{C}})[[\lambda]]$.

From (60) we already see that the behaviour of $\boldsymbol{\iota}^*_\kappa$ and $\boldsymbol{h}^{(\kappa)}$ is more complicated under complex conjugation. For the relevant geometric series in (35) we have the following result:

Lemma 4.4 *Let $f \in \mathcal{C}^\infty(M)[[\lambda]]$. Then the operators*

$$A^a_\kappa(f) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{k} \left(\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0\right)^{k-\ell} \mathrm{i}(e^a) h_0 \overline{\left(\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0\right)^{\ell-1} \overline{f}} \tag{65}$$

and

$$B_\kappa(f) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{k} \left(\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0\right)^{k-\ell} \mathrm{i}(\Delta) h_0 \overline{\left(\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0\right)^{\ell-1} \overline{f}} \tag{66}$$

yield well-defined $\mathbb{C}[[\lambda]]$-linear maps $A^a_\kappa, B_\kappa : \mathcal{C}^\infty(M)[[\lambda]] \longrightarrow \mathcal{C}^\infty(M)[[\lambda]]$ such that

$$\overline{\left(\mathsf{id} + \left(\boldsymbol{\partial}_1^{(\kappa)} - \partial_1\right) h_0\right)^{-1} f} = \left(\mathsf{id} + \left(\boldsymbol{\partial}_1^{(\kappa)} - \partial_1\right) h_0\right)^{-1} \overline{f} + \mathrm{i}\lambda \mathscr{L}_{(e_a)_M} A^a(\overline{f}) + \mathrm{i}\lambda(\overline{\kappa} + \kappa) B(\overline{f}) \tag{67}$$

$$= \left(\mathsf{id} + \left(\boldsymbol{\partial}_1^{(\kappa)} - \partial_1\right) h_0\right)^{-1} \overline{f} + \mathrm{i}\lambda A^a\left(\mathscr{L}_{(e_a)_M}(\overline{f})\right) + \mathrm{i}\lambda(\overline{\kappa} + \kappa - 1) B(\overline{f}) \tag{68}$$

Proof. Since the term with the two insertions does not contribute and since ad_{e_a} vanishes on functions we get from (63)

$$\overline{\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0 f} = \left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0 \overline{f} - \mathrm{i}\lambda \varrho(e_a)\, \mathrm{i}(e^a) h_0 \overline{f} + \mathrm{i}\lambda(\overline{\kappa} + \kappa)\, \mathrm{i}(\Delta) h_0 \overline{f}. \qquad (*)$$

Now using (64) we get the alternative version

$$\overline{\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0 f} = \left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0 \overline{f} - \mathrm{i}\lambda\, \mathrm{i}(e^a) h_0 (\varrho(e_a)\overline{f}) + \mathrm{i}\lambda(\overline{\kappa} + \kappa - 1)\, \mathrm{i}(\Delta) h_0 \overline{f} \qquad (**)$$

for the commutation relation. It is this equation which motivates $\kappa = \frac{1}{2}$ instead of $\kappa = 0$. Applying the general commutation relation $CB^k = [C, B]B^{k-1} + B[C, B]B^{k-2} + \cdots + B^{k-1}[C, B] + B^k C$ to the complex conjugation and the map $\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0$ gives

$$\overline{\left(\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0\right)^k f} = \left(\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0\right)^k \overline{f}$$
$$+ \sum_{\ell=1}^{k} \left(\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0\right)^{k-\ell} (-\mathrm{i}\lambda\varrho(e_a)\, \mathrm{i}(e^a) h_0 + \mathrm{i}\lambda(\overline{\kappa} + \kappa)\, \mathrm{i}(\Delta) h_0)\, \overline{\left(\left(\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}\right) h_0\right)^{\ell-1} f}.$$

Using the operators A^a_κ and B_κ, which are clearly well-defined as formal series since the difference $\partial_1 - \boldsymbol{\partial}_1^{(\kappa)}$ is at least of order λ, we get for the geometric series

$$\overline{\left(\mathsf{id} + \left(\boldsymbol{\partial}_1^{(\kappa)} - \partial_1\right) h_0\right)^{-1} f} = \left(\mathsf{id} + \left(\boldsymbol{\partial}_1^{(\kappa)} - \partial_1\right) h_0\right)^{-1} \overline{f} - \mathrm{i}\lambda\varrho(e_a) A^a_\kappa(\overline{f}) + \mathrm{i}\lambda(\overline{\kappa} + \kappa) B_\kappa(\overline{f}),$$

since the action of e_a can be commuted to the *left* as all operators are G-invariant. This proves the first equation (67) since $A^a_\kappa(\overline{f})$ is a function whence the left action of e_a is just $-\mathscr{L}_{(e_a)_M}$. Now conversely, using the second version $(**)$ we can commute the action of e_a to the right, as now only invariant operators remain. This gives (68). □

Corollary 4.5 *Let $f \in \mathcal{C}^\infty(M)[[\lambda]]$. Then we have*

$$\overline{\boldsymbol{\iota}^*_\kappa f} = \boldsymbol{\iota}^*_\kappa \overline{f} + \mathrm{i}\lambda \mathscr{L}_{(e_a)_C}\, \iota^* A^a_\kappa(\overline{f}) + \mathrm{i}\lambda(\overline{\kappa} + \kappa)\iota^* B_\kappa(\overline{f}) \qquad (69)$$

$$= \boldsymbol{\iota}^*_\kappa \overline{f} + \mathrm{i}\lambda\iota^* A^a_\kappa \left(\mathscr{L}_{(e_a)_M} \overline{f}\right) + \mathrm{i}\lambda(\overline{\kappa} + \kappa - 1)\iota^* B_\kappa(\overline{f}). \qquad (70)$$

Analogously, we obtain the behaviour of the homotopy $\boldsymbol{h}^{(\kappa)}$ under complex conjugation. It is clear from (70) that the value

$$\kappa = \frac{1}{2} \qquad (71)$$

will simplify things drastically as in this case the presence of the operator B is absent in (70). Thus from now on we will exclusively consider $\kappa = \frac{1}{2}$ and omit the subscript κ at all relevant places. A first consequence of this choice is the following:

Corollary 4.6 *If $f \in \mathcal{C}^\infty(M)[[\lambda]]$ is G-invariant then*

$$\overline{\boldsymbol{\iota}^* f} = \boldsymbol{\iota}^* \overline{f}. \qquad (72)$$

Even though it is not in the main line of our argument according to the previous section, we can now directly prove that $\star_{\mathrm{red}}$ is Hermitian:

Proposition 4.7 *The star product $\star_{\mathrm{red}}$ is Hermitian.*

Proof. Let $u, v \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$. Then we have

$$\overline{\pi^*(u \star_{\mathrm{red}} v)} = \overline{\boldsymbol{\iota}^* \left(\mathrm{prol}(\pi^* u) \star \mathrm{prol}(\pi^* v)\right)} = \boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\pi^* u) \star \mathrm{prol}(\pi^* v)}\right) = \boldsymbol{\iota}^* \left(\mathrm{prol}(\pi^* \overline{v}) \star \mathrm{prol}(\pi^* \overline{u})\right)$$

$$= \pi^*(\overline{v} \star_{\mathrm{red}} \overline{u}),$$

since by the G-invariance of prol and $\star$ as in (29) we know that $\mathrm{prol}(\pi^* u) \star \mathrm{prol}(\pi^* v)$ is G-invariant and thus Corollary 4.6 applies. □

Remark 4.8 From the proof we see that one needs the complex conjugation of the functions $\mathcal{C}^\infty(C)[[\lambda]]$. In a purely algebraic setting as in Section 4.1 this would mean to have a *-involution on the *module* $\mathcal{A}/\mathcal{J}$, which is clearly a non-canonical extra structure. Thus also the above seemingly canonical proof that $\star_{\mathrm{red}}$ is Hermitian is not that conceptual from the point of view of our considerations in Section 4.1.

We will now come back to the construction of the positive functional. First we choose a formal series of densities $\mu = \sum_{r=0}^\infty \lambda^r \mu_r \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*C)[[\lambda]]$ on C such that $\overline{\mu} = \mu$ is real and $\mu_0 > 0$ is everywhere positive. Moreover, we require that μ transforms under the G action as follows

$$\mathsf{L}^*_{g^{-1}} \mu = \frac{1}{\Delta(g)} \mu, \tag{73}$$

where $\Delta : G \longrightarrow \mathbb{R}^+$ is the modular function of G and we take the *left* action of G on densities. Recall that Δ is the Lie group homomorphism obtained from exponentiating the Lie algebra homomorphism $\Delta(\xi) = \mathsf{tr}\, ad_\xi$, thereby motivating our notation. For the (well-known) existence of densities with (73) see Appendix A.

Definition 4.9 *For $\phi, \psi \in \mathcal{C}^\infty_0(C)[[\lambda]]$ we define*

$$\langle \phi, \psi \rangle_\mu = \int_C \boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi)\right) \mu. \tag{74}$$

Lemma 4.10 *Let $\phi, \psi \in \mathcal{C}^\infty_0(C)[[\lambda]]$ and $f \in \mathcal{C}^\infty_0(M)[[\lambda]]$. Then $\langle \phi, \psi \rangle_\mu$ is well-defined and we have*

$$\langle \phi, \psi \rangle_\mu = \int_C \left(\overline{\mathrm{prol}(\phi)} \bullet \psi\right) \mu \tag{75}$$

and

$$\overline{\int_C \boldsymbol{\iota}^*(f) \mu} = \int_C \boldsymbol{\iota}^* \left(\overline{f}\right) \mu. \tag{76}$$

Proof. Even though the prolongation is not a local operator the supports are only changed in transverse directions to C. It follows that the support of the integrand in (75) as well as in (76) is compact in every order of λ. Then the first part is clear from (42). For the second we compute

$$\begin{aligned}
\int_C \overline{\boldsymbol{\iota}^* f} \mu &= \int_C \boldsymbol{\iota}^* \overline{f} + \mathrm{i}\lambda \int_C \mathscr{L}_{(e_a)_C}\, \iota^* A^a(\overline{f}) \mu + \mathrm{i}\lambda \int_C \iota^* B(\overline{f}) \mu \\
&= \int_C \boldsymbol{\iota}^* \overline{f} - \mathrm{i}\lambda \int_C \iota^* A^a(\overline{f})\, \mathscr{L}_{(e_a)_C} \mu + \mathrm{i}\lambda \int_C \iota^* B(\overline{f}) \mu \\
&= \int_C \boldsymbol{\iota}^* \overline{f} - \Delta(e_a) \mathrm{i}\lambda \int_C \iota^* A^a(\overline{f}) \mu + \mathrm{i}\lambda \int_C \iota^* B(\overline{f}) \mu.
\end{aligned}$$

Here we used $\mathscr{L}_{\xi_C} \mu = \Delta(\xi) \mu$ which follows from differentiating (73). Finally, a close look at the definitions of A^a and B shows that $\Delta(e_a) A^a(f) = B(f)$. This shows the second part. □

Proposition 4.11 *The scalar product $\langle\cdot,\cdot\rangle_\mu$ makes $\mathcal{C}_0^\infty(C)[[\lambda]]$ a pre Hilbert space over $\mathbb{C}[[\lambda]]$ and the left module structure $\bullet$ becomes a *-representation of $(\mathcal{C}^\infty(M)[[\lambda]],\star)$ on it.*

Proof. The $\mathbb{C}[[\lambda]]$-linearity of $\langle\cdot,\cdot\rangle_\mu$ in the second argument is clear. For the symmetry we compute

$$\overline{\langle\phi,\psi\rangle_\mu} = \overline{\int_C \boldsymbol{\iota}^*\left(\overline{\mathrm{prol}(\phi)}\star\mathrm{prol}(\psi)\right)\mu} \overset{(76)}{=} \int_C \boldsymbol{\iota}^*\left(\overline{\overline{\mathrm{prol}(\phi)}\star\mathrm{prol}(\psi)}\right)\mu = \int_C \boldsymbol{\iota}^*\left(\overline{\mathrm{prol}(\psi)}\star\mathrm{prol}(\phi)\right)\mu$$
$$= \langle\psi,\phi\rangle_\mu\,,$$

using that $\star$ is Hermitian. Finally, for $\phi=\sum_{r=r_0}\lambda^r\phi_r$ with $\phi_{r_0}\neq 0$ the lowest non-vanishing term in $\langle\phi,\phi\rangle_\mu$ is simply given by the μ_0-integral of $\overline{\phi_{r_0}}\phi_{r_0}$ over C which gives a positive result since μ_0 is a *positive* density. Thus $\mathcal{C}_0^\infty(C)[[\lambda]]$ becomes a pre Hilbert space indeed. Now for $f\in\mathcal{C}^\infty(M)[[\lambda]]$ we compute

$$\begin{aligned}
\langle\phi, f\bullet\psi\rangle_\mu &\overset{(75)}{=} \int_C \left(\overline{\mathrm{prol}(\phi)}\bullet(f\bullet\psi)\right)\mu\\
&= \int_C \left((\overline{\mathrm{prol}(\phi)}\star f)\bullet\psi\right)\mu\\
&= \int_C \left((\overline{\overline{f}\star\mathrm{prol}(\phi)})\bullet\psi\right)\mu\\
&= \int_C \boldsymbol{\iota}^*\left((\overline{\overline{f}\star\mathrm{prol}(\phi)})\star\mathrm{prol}(\psi)\right)\mu\\
&\overset{(76)}{=} \overline{\int_C \boldsymbol{\iota}^*\left(\overline{\mathrm{prol}(\psi)}\star(\overline{f}\star\mathrm{prol}(\phi))\right)\mu}\\
&= \overline{\int_C \boldsymbol{\iota}^*\left((\overline{\mathrm{prol}(\psi)}\star\overline{f})\star\mathrm{prol}(\phi)\right)\mu}\\
&= \overline{\int_C \left((\overline{\mathrm{prol}(\psi)}\star\overline{f})\bullet\phi\right)\mu}\\
&= \overline{\int_C \left(\overline{\mathrm{prol}(\psi)}\bullet(\overline{f}\bullet\phi)\right)\mu}\\
&\overset{(75)}{=} \overline{\langle\psi,\overline{f}\bullet\phi\rangle_\mu},
\end{aligned}$$

using the fact that $\bullet$ is a left $\star$-module structure. It follows that we have a *-representation. □

We will now show that this *-representation is actually the GNS representation of the following positive linear functional:

Definition 4.12 *For $f\in\mathcal{C}_0^\infty(M)[[\lambda]]$ we define the $\mathbb{C}[[\lambda]]$-linear functional ω_μ by*

$$\omega_\mu(f) = \int_C \boldsymbol{\iota}^*(f)\mu. \tag{77}$$

First note that $\boldsymbol{\iota}^*f\in\mathcal{C}_0^\infty(C)[[\lambda]]$ by Lemma 3.5 whence ω_μ is well-defined. Since neither C nor M need to be compact we are dealing with a *-algebra $\mathcal{C}_0^\infty(M)[[\lambda]]$ *without* unit in general. The following lemma would be much easier if $1\in\mathcal{C}_0^\infty(C)$:

Lemma 4.13 *The* $\mathbb{C}[[\lambda]]$*-linear functional* ω_μ *is positive and its Gel'fand ideal is*

$$\mathcal{J}_{\omega_\mu} = \left\{ f \in \mathcal{C}_0^\infty(M)[[\lambda]] \;\middle|\; \boldsymbol{\iota}^* f = 0 \right\}. \tag{78}$$

Proof. In order to show the positivity $\omega_\mu(\overline{f} \star f) \geq 0$ it is sufficient to consider $f \in \mathcal{C}_0^\infty(M)$ without higher λ-orders thanks to [27, Prop. 7.1.51]. For $f \in \mathcal{C}_0^\infty(M)$ we choose a $\chi = \overline{\chi} \in \mathcal{C}_0^\infty(C)$ such that $\operatorname{prol}(\chi)\big|_{\operatorname{supp} f} = 1$ which is clearly possible. By the locality of $\star$ we get $f \star \operatorname{prol}(\chi) = f = \operatorname{prol}(\chi) \star f$ and hence also $f \bullet \chi = \boldsymbol{\iota}^* f$. Thus

$$\begin{aligned} \langle \chi, f \bullet \chi \rangle_\mu &= \int_C \left(\overline{\operatorname{prol}(\chi)} \bullet (f \bullet \chi) \right) \mu = \int_C \left((\operatorname{prol}(\chi) \star f) \bullet \chi \right) \mu = \int_C (f \bullet \chi) \mu = \int_C \boldsymbol{\iota}^*(f) \mu \\ &= \omega_\mu(f). \end{aligned}$$

Since $\operatorname{supp}(\overline{f} \star f) \subseteq \operatorname{supp} f$ the same applies for $\overline{f} \star f$ and we have

$$\omega_\mu(\overline{f} \star f) = \left\langle \chi, (\overline{f} \star f) \bullet \chi \right\rangle_\mu = \langle f \bullet \chi, f \bullet \chi \rangle_\mu = \langle \boldsymbol{\iota}^* f, \boldsymbol{\iota}^* f \rangle_\mu \geq 0$$

by the positivity of $\langle \cdot, \cdot \rangle_\mu$. Finally, $\omega_\mu(\overline{f} \star f) = 0$ iff $\boldsymbol{\iota}^* f = 0$ is clear from this computation. □

This lemma allows to identify the GNS representation induced by ω_μ easily. Recall that the GNS representation automatically extends to the whole algebra since $\mathcal{C}_0^\infty(M)[[\lambda]]$ is a *-ideal, see e.g. [27, Lem. 7.2.18]:

Theorem 4.14 *The GNS representation of* $\mathcal{C}^\infty(M)[[\lambda]]$ *on* $\mathcal{H}_{\omega_\mu} = \mathcal{C}_0^\infty(M)[[\lambda]] / \mathcal{J}_{\omega_\mu}$ *is unitarily equivalent to the* **-representation* $\bullet$ *on* $\mathcal{C}_0^\infty(C)[[\lambda]]$ *where the inner product is* $\langle \cdot, \cdot \rangle_\mu$*. The unitary intertwiner is explicitly given by*

$$\mathcal{C}_0^\infty(M)[[\lambda]] / \mathcal{J}_{\omega_\mu} \ni \psi_f \mapsto \boldsymbol{\iota}^* f \in \mathcal{C}_0^\infty(C)[[\lambda]]. \tag{79}$$

Proof. By (78) it follows that (79) is well-defined and injective. Now let $\chi \in \mathcal{C}^\infty(M)$ be a function such that χ is equal to one in an open neighbourhood of C but has compact support in directions of the fibers of the tubular neighbourhood of C. Clearly, such a function exists (and can even be chosen to be G-invariant thanks to the properness of the action). It follows that $\chi \operatorname{prol}(\phi) \in \mathcal{C}_0^\infty(M)[[\lambda]]$ for $\phi \in \mathcal{C}_0^\infty(C)[[\lambda]]$. Moreover, the locality of $\boldsymbol{\iota}^*$ according to Lemma 3.5 shows that $\boldsymbol{\iota}^*(\chi \operatorname{prol}(\phi)) = \phi$ proving the surjectivity of (79). Finally, let $f, g \in \mathcal{C}_0^\infty(M)$ and chose $\chi \in \mathcal{C}_0^\infty(C)$ such that $\operatorname{prol}(\chi)\big|_{\operatorname{supp} f \cup \operatorname{supp} g} = 1$. Again, such a χ exists. Then we can proceed as in the proof of Lemma 4.13 and have

$$\langle \boldsymbol{\iota}^* f, \boldsymbol{\iota}^* g \rangle_\mu = \langle f \bullet \chi, g \bullet \chi \rangle_\mu = \left\langle \chi, (\overline{f} \star g) \bullet \chi \right\rangle_\mu = \omega_\mu(\overline{f} \star g) = \langle \psi_f, \psi_g \rangle_{\omega_\mu}.$$

This shows that (79) is isometric on ψ_f, ψ_g without higher orders of λ. By $\mathbb{C}[[\lambda]]$-sesquilinearity of both inner products this holds in general whence (79) is unitary. But then by (40)

$$\pi_{\omega_\mu}(f)\psi_g \mapsto \boldsymbol{\iota}^*(f \star g) = \boldsymbol{\iota}^* \left(f \star \operatorname{prol}(\boldsymbol{\iota}^* g) + f \star \boldsymbol{\partial}_1(\boldsymbol{h}_0 g) \right) = f \bullet \boldsymbol{\iota}^* g + \boldsymbol{\iota}^* \left(\boldsymbol{\partial}_1(f \star \boldsymbol{h}_0 g) \right) = f \bullet \boldsymbol{\iota}^* g,$$

since $\boldsymbol{\iota}^* \boldsymbol{\partial}_1 = 0$. Thus (79) intertwines the GNS representation π_{ω_μ} into $\bullet$ as claimed. □

Remark 4.15 (Quantization of coisotropic submanifolds) In [5] as well as in [12, 13] the question was raised whether the classical $\mathcal{C}^\infty(M)$-module structure of $\mathcal{C}^\infty(C)$ of a submanifold $\iota : C \longrightarrow M$ can be quantized with respect to a given star product. In general, C has to be coisotropic but there are still obstructions beyond this zeroth order condition, see e.g. [29] for a simple counter-example. In view of Theorem 4.14 one can rephrase and sharpen this task as follows: one should try to find a positive density $\mu_0 > 0$ on C such that the functional $\omega_0(f) = \int_C f \mu_0$

allows for a deformation into a positive functional with respect to $\star$ *and* such that it yields a GNS pre Hilbert space isomorphic to $\mathcal{C}_0^\infty(C)[[\lambda]]$. This way one could obtain a deformation of the classical module structure which is even a *-representation. Note that in the zeroth order this is consistent: the Gel'fand ideal of the classical integration functional is precisely the vanishing ideal of C. Note also, that *every* classically positive functional can be deformed into a positive functional with respect to $\star$. However, the behaviour of the Gel'fand ideal under this deformation is rather mysterious, see e.g. the discussion in [27, Sect. 7.1.5 & Sect. 7.2.4] for further details and references. In any case, Theorem 4.14 gives some hope that this might be a reasonable approach also in some greater generality.

4.3 The reduced *-involution

According to Proposition 4.1 we have to show that the right $\bullet_{\mathrm{red}}$-multiplications are adjointable with respect to $\langle\cdot,\cdot\rangle_\mu$. We prove a slightly more general statement:

Lemma 4.16 *Let P_r be bidifferential operators on C and let $\mu_0 > 0$ be an everywhere positive, smooth density on C.*

i.) Then the inner product

$$\langle\phi,\psi\rangle_P = \int_C \left(\overline{\phi}\psi + \sum_{r=1}^{\infty}\lambda^r P_r(\overline{\phi},\psi)\right)\mu_0 \tag{80}$$

is well-defined for $\phi,\psi \in \mathcal{C}_0^\infty(C)[[\lambda]]$ and non-degenerate.

ii.) Every formal series $D = \sum_{r=0}^{\infty}\lambda^r D_r$ of differential operators on C is adjointable with respect to $\langle\cdot,\cdot\rangle_P$.

iii.) The adjoint D^+ is again a formal series of differential operators and D_0^+ coincides with the usual adjoint of D_0 with respect to the integration density μ_0.

Proof. The first part is clear. The second is shown order by order. Assume that we have found a differential operator $D^+_{(k)} = D_0^+ + \lambda D_1^+ + \cdots + \lambda^k D_k^+$ such that

$$\langle\phi, D\psi\rangle_P - \left\langle D^+_{(k)}\phi,\psi\right\rangle_P = \int_C \left(\sum_{r=k+1}^{\infty}\lambda^r E_r^{(k)}(\overline{\phi},\psi)\right)\mu_0$$

with some bidifferential operators $E_r^{(k)}$. For $k=0$ this is clearly achievable by the choice of D_0^+ as claimed in the third part. For a differential operator D^+_{k+1} we have

$$\begin{aligned}
&\langle\phi, D\psi\rangle_P - \left\langle\left(D^+_{(k)} + \lambda^{k+1}D^+_{k+1}\right)\phi,\psi\right\rangle_P \\
&\quad= \lambda^{k+1}\int_C\left(E^{(k)}_{k+1}(\overline{\phi},\psi) - \overline{D^+_{k+1}\phi}\psi\right)\mu_0 + \sum_{r=k+2}^{\infty}\lambda^r\int_C\left(E_r^{(k)}(\overline{\phi},\psi) + P_{r-(k+1)}\left(\overline{D^+_{k+1}\phi},\psi\right)\right)\mu_0.
\end{aligned}$$

Integration by parts shows that we can arrange that all derivatives in $E^{(k)}_{k+1}(\overline{\phi},\psi)$ are moved to $\overline{\phi}$ including terms coming from derivatives of μ_0. But then we can chose D^+_{k+1} to cancel the order λ^{k+1}. Since with this choice, D^+_{k+1} is a differential operator itself, the error terms in higher orders are encoded by bidifferential operators again. Thus we can proceed by induction showing the second part. The third is clear from this construction. □

Theorem 4.17 (Reduced *-Involution) *Let $u \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$. Then there exists a unique $u^* \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ such that for all $\phi,\psi \in \mathcal{C}_0^\infty(C)[[\lambda]]$*

$$\langle\phi,\psi\bullet_{\mathrm{red}} u\rangle_\mu = \langle\phi\bullet_{\mathrm{red}} u^*,\psi\rangle_\mu. \tag{81}$$

The map $u \mapsto u^*$ is a *-involution for $\star_{\mathrm{red}}$ of the form

$$u^* = \overline{u} + \sum_{r=1}^{\infty} \overline{I_r(u)} \tag{82}$$

with differential operators I_r on M_{red}.

Proof. From Proposition 3.11, *iv.)* we know that the map $\phi \mapsto \phi \bullet_{\mathrm{red}} u$ is a formal series of differential operators. Moreover, from the locality of $\star$ and $\boldsymbol{\iota}^*$ it is clear that $\boldsymbol{\iota}^*(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}\,\psi) = \overline{\phi}\psi + \sum_{r=1}^{\infty} \lambda^r \tilde{P}_r(\overline{\phi}, \psi)$ with some bidifferential operators $\tilde{P}_r$. Since $\mu_0 > 0$ we can write $\mu = \frac{\mu}{\mu_0}\mu_0$ with some *function* $\frac{\mu}{\mu_0} = 1 + \cdots \in \mathcal{C}^\infty(C)[[\lambda]]$. Resorting by powers of λ we conclude that $\langle \cdot, \cdot \rangle_\mu$ is of the form (80). Then Lemma 4.16, *ii.)* shows that $\phi \mapsto \phi \bullet_{\mathrm{red}} u$ is adjointable. By Proposition 4.1 we known that the adjoint is necessarily of the form $\phi \mapsto \phi \bullet_{\mathrm{red}} u^*$ with a unique $u^* \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$. Moreover, it is clear that $u \mapsto u^*$ is a *-involution for $\star_{\mathrm{red}}$. Then $u^* = \overline{u} + \cdots$ follows from Lemma 4.16, *iii.)*. Finally, since the construction of the adjoint as in Lemma 4.16 consists in finitely many integrations by parts and multiplications by coefficient functions in each fixed order of λ, we conclude that the higher order corrections in (82) are differential. □

We want to determine the *-involution (82) more closely and relate it to the complex conjugation, which is a *-involution of $\star_{\mathrm{red}}$ as well, see Proposition 4.7. To this end we consider the formal series of densities $\Omega \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M_{\mathrm{red}})[[\lambda]]$ corresponding to μ under the canonical isomorphism (161). To proceed locally, we chose a small enough open subset $U \subseteq M_{\mathrm{red}}$ and a trivialization $\Phi : U \times G \longrightarrow \pi^{-1}(U) \subseteq C$ where we trivialize the principal bundle C as a *right* principal bundle, i.e. Φ is equivariant for the right actions.

Proposition 4.18 *Let $\Omega \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M_{\mathrm{red}})[[\lambda]]$ be the pre-image of μ under* (161).

i.) One has $\Omega_0 > 0$, $\overline{\Omega} = \Omega$ and locally $\Phi^*\left(\mu\big|_{\pi^{-1}(U)}\right) = \Omega\big|_U \boxtimes \mathrm{d}^{\mathrm{left}}\, g$.

ii.) For $u, v \in \mathcal{C}_0^\infty(M_{\mathrm{red}})[[\lambda]]$ one has

$$\int_{M_{\mathrm{red}}} v \star_{\mathrm{red}} u \; \Omega = \int_{M_{\mathrm{red}}} \overline{u^*} \star_{\mathrm{red}} v \; \Omega. \tag{83}$$

Proof. The first part is clear from (163) discussed in the appendix. To prove (83) it is clearly sufficient to assume $u, v \in \mathcal{C}_0^\infty(U)[[\lambda]]$ by a partition of unity argument. Choose a $\chi \in \mathcal{C}_0^\infty(G)$ with $\int_G \chi \, \mathrm{d}^{\mathrm{left}}\, g = 1$. We use now the trivialization $\Phi : U \times G \longrightarrow \pi^{-1}(U)$ to identify functions on $U \times G$ with those on $\pi^{-1}(U) \subseteq C$ without explicitly writing Φ^* to simplify our notation. Then we consider $(1 \otimes \chi) \bullet_{\mathrm{red}} v \in \mathcal{C}_0^\infty(U \times G)[[\lambda]]$. Moreover, we use $1 \bullet_{\mathrm{red}} u = \pi^* u$ according to Proposition 3.11, *vi.)*. This allows to evaluate the inner product $\left\langle \overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v}, \pi^* u \right\rangle_\mu$ in two ways. First we have

$$\begin{aligned}
\left\langle \overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v}, \pi^* u \right\rangle_\mu &= \int_C \boldsymbol{\iota}^* \left(\overline{\mathrm{prol}\left(\overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v}\right)} \star \mathrm{prol}(\pi^* u) \right) \mu \\
&= \int_C ((1 \otimes \chi) \bullet_{\mathrm{red}} v) \bullet_{\mathrm{red}} u \; \mu \\
&= \int_C (1 \otimes \chi) \bullet_{\mathrm{red}} (v \star_{\mathrm{red}} u) \; \mu \\
&= \int_{U \times G} (1 \otimes \chi) \bullet_{\mathrm{red}} (v \star_{\mathrm{red}} u) \; \Omega \boxtimes \mathrm{d}^{\mathrm{left}}\, g \\
&= \int_U \left(p \mapsto \int_G \mathsf{L}^*_{g^{-1}} \left((1 \otimes \chi) \bullet_{\mathrm{red}} (v \star_{\mathrm{red}} u) \right) \mathrm{d}^{\mathrm{left}}\, g \Big|_{(p,e)} \right) \Omega
\end{aligned}$$

$$\overset{(54)}{=} \int_U \left(p \mapsto \int_G \mathsf{L}^*_{g^{-1}}(1 \otimes \chi)\, \mathrm{d}^{\mathrm{left}} g \right) \bullet_{\mathrm{red}} (v \star_{\mathrm{red}} u)\Big|_{(p,e)} \Omega$$

$$= \int_U (1 \bullet_{\mathrm{red}} (v \star_{\mathrm{red}} u))\Big|_{(p,e)} \Omega$$

$$= \int_U v \star_{\mathrm{red}} u \, \Omega,$$

which is the left hand side of (83). Note that the integral $\int_G \mathsf{L}^*_{g^{-1}}(1 \otimes \chi)\, \mathrm{d}^{\mathrm{left}} g$ is still understood as a function on $U \times G$. The second way to compute the inner product is

$$\left\langle \overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v}, \pi^* u \right\rangle_\mu = \left\langle \overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v}, 1 \bullet_{\mathrm{red}} u \right\rangle_\mu$$

$$\overset{(*)}{=} \left\langle \overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v} \bullet_{\mathrm{red}} u^*, 1 \right\rangle_\mu$$

$$= \int_C \boldsymbol{\iota}^* \left(\overline{\mathrm{prol}\left(\overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v} \bullet_{\mathrm{red}} u^* \right)} \star 1 \right) \mu$$

$$= \int_C \overline{\overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v} \bullet_{\mathrm{red}} u^*} \, \mu$$

$$= \int_C \overline{\boldsymbol{\iota}^* \left(\mathrm{prol}\left(\overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v} \right) \star \mathrm{prol}(\pi^* u^*) \right)} \mu$$

$$\overset{(76)}{=} \int_C \boldsymbol{\iota}^* \left(\overline{\mathrm{prol}\left(\overline{(1 \otimes \chi) \bullet_{\mathrm{red}} v} \right) \star \mathrm{prol}(\pi^* u^*)} \right) \mu$$

$$= \int_C \boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\pi^* u^*)} \star \mathrm{prol}\left((1 \otimes \chi) \bullet_{\mathrm{red}} v \right) \right) \mu$$

$$= \int_U \left(p \mapsto \int_G \mathsf{L}^*_{g^{-1}} \left(\boldsymbol{\iota}^* \left(\mathrm{prol}(\pi^* \overline{u^*}) \star \mathrm{prol}\left((1 \otimes \chi) \bullet_{\mathrm{red}} v \right) \right) \right) \mathrm{d}^{\mathrm{left}} g \Big|_{(p,e)} \right) \Omega$$

$$\overset{(54)}{=} \int_U \left(p \mapsto \boldsymbol{\iota}^* \left(\mathrm{prol}(\pi^* \overline{u^*}) \star \mathrm{prol}\left(\left(1 \otimes \int_G \mathsf{L}^*_{g^{-1}} \chi \, \mathrm{d}^{\mathrm{left}} g \right) \bullet_{\mathrm{red}} v \right) \right) \Big|_{(p,e)} \right) \Omega$$

$$= \int_U \left(p \mapsto \boldsymbol{\iota}^* \left(\mathrm{prol}(\pi^* \overline{u^*}) \star \mathrm{prol}(\pi^* v) \right) \Big|_{(p,e)} \right) \Omega$$

$$= \int_U \overline{u^*} \star_{\mathrm{red}} v \, \Omega,$$

where we have used in $(*)$ that the integrals are still well-defined even if one of the functions is not in $\mathcal{C}^\infty_0(C)[[\lambda]]$ but the other is. Moreover, we used the fact that all maps are G-equivariant. Thus the integral can be moved directly in front of χ where it gives the constant function 1 on $U \times G$. Then $1 \bullet_{\mathrm{red}} v = \pi^* v$ can be applied once more. □

Remark 4.19 (KMS functional) Since we already know that the complex conjugation is a *-involution for $\star_{\mathrm{red}}$ as well, the map

$$I_\mu : u \mapsto \overline{u^*} \tag{84}$$

is a $\mathbb{C}[[\lambda]]$-linear *automorphism* of $\star_{\mathrm{red}}$. Then the result (83) means that the functional

$$\tau_\Omega : \mathcal{C}^\infty_0(M_{\mathrm{red}})[[\lambda]] \ni u \;\mapsto\; \tau_\Omega(u) = \int_{M_{\mathrm{red}}} u \, \Omega \in \mathbb{C}[[\lambda]] \tag{85}$$

is actually a *KMS functional* with respect to the I_μ, i.e. we have the KMS property

$$\tau_\Omega(v \star_{\mathrm{red}} u) = \tau_\Omega(I_\mu(u) \star_{\mathrm{red}} v), \tag{86}$$

see [3] as well as [27, Sect. 7.1.4] for a discussion of the KMS condition in the context of deformation quantization including a proof of the classification of KMS functionals.

The construction of the *-involution * depends on the choice of μ. Two such choices μ and μ' are related by a unique function $\varrho = \overline{\varrho} \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ with $\varrho_0 > 0$ via $\mu' = \pi^*\varrho\mu$. The corresponding densities on M_{red} are then related by $\Omega' = \varrho\Omega$. To relate the *-involutions * and $^{*'}$ corresponding to μ and μ', respectively, we consider the KMS functionals τ_Ω and $\tau_{\Omega'}$:

Lemma 4.20 *Let $\Omega' = \varrho\Omega$ be as above. Then there exists a unique $\boldsymbol{\varrho} \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ with $\boldsymbol{\varrho}_0 = \varrho_0 > 0$ such that for all $u \in \mathcal{C}^\infty_0(M_{\mathrm{red}})[[\lambda]]$ we have*

$$\tau_{\Omega'}(u) = \tau_\Omega(\boldsymbol{\varrho} \star_{\mathrm{red}} u). \tag{87}$$

Proof. This is shown inductively order by order in λ. Clearly, $\boldsymbol{\varrho}_0 = \varrho_0$ is the unique choice to satisfy (87) in zeroth order. Then $\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, \ldots$ are obtained by integration by parts, relying on the fact that $\star_{\mathrm{red}}$ is bidifferential. Uniqueness is clear from the non-degeneracy of the integration. □

Theorem 4.21 *The *-involutions * and $^{*'}$ obtained from different choices of μ and μ', respectively, are related by an inner automorphism*

$$u^{*'} = \overline{\boldsymbol{\varrho}} \star_{\mathrm{red}} u^* \star_{\mathrm{red}} \overline{\boldsymbol{\varrho}}^{-1}, \tag{88}$$

with $\boldsymbol{\varrho}$ as in Lemma 4.20.

Proof. This is an easy computation. For $u, v \in \mathcal{C}^\infty_0(M_{\mathrm{red}})[[\lambda]]$ we have

$$\begin{aligned}
\tau_\Omega\left(\boldsymbol{\varrho} \star_{\mathrm{red}} \overline{u^{*'}} \star_{\mathrm{red}} v\right) &= \tau_{\Omega'}\left(\overline{u^{*'}} \star_{\mathrm{red}} v\right) \\
&\overset{(86)}{=} \tau_{\Omega'}\left(v \star_{\mathrm{red}} u\right) \\
&= \tau_\Omega\left(\boldsymbol{\varrho} \star_{\mathrm{red}} v \star_{\mathrm{red}} u\right) \\
&\overset{(86)}{=} \tau_\Omega\left(\overline{u^*} \star_{\mathrm{red}} \boldsymbol{\varrho} \star_{\mathrm{red}} v\right),
\end{aligned}$$

from which we deduce $\boldsymbol{\varrho} \star_{\mathrm{red}} \overline{u^{*'}} = \overline{u^*} \star_{\mathrm{red}} \boldsymbol{\varrho}$ as v is arbitrary. Since $\boldsymbol{\varrho}$ starts with $\boldsymbol{\varrho}_0 = \varrho_0 > 0$ it is $\star_{\mathrm{red}}$-invertible. This completes the proof. □

Corollary 4.22 *The *-involution * coincides with the complex conjugation iff μ yields a trace density Ω, i.e. τ_Ω is a trace functional.*

Proof. If $u^* = \overline{u}$ then Proposition 4.18 gives the trace property immediately. Conversely, assume τ_Ω is a trace. Then (86) implies $\tau_\Omega(\overline{u^*} \star_{\mathrm{red}} v) = \tau_\Omega(u \star_{\mathrm{red}} v)$ for all $u, v \in \mathcal{C}^\infty_0(M_{\mathrm{red}})[[\lambda]]$. But this gives $\overline{u^*} = u$ by the non-degeneracy of the integration. □

Remark 4.23 (Unimodular Poisson structures) The *existence* of a trace density for $\star_{\mathrm{red}}$ is non-trivial: the lowest order condition implies that Ω_0 is a Poisson trace, i.e. the functional τ_{Ω_0} vanishes on Poisson brackets. Thus the existence of such a Ω_0 is equivalent to say that the Poisson structure of M_{red} is *unimodular*, see e.g. [28].

Remark 4.24 (Symplectic trace density) In the case where M_{red} is symplectic the Liouville volume density $\Omega_0 = |\omega_{\mathrm{red}} \wedge \cdots \wedge \omega_{\mathrm{red}}| \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M_{\mathrm{red}})$ is known to be (up to a normalization constant) the unique Poisson trace density. Moreover, in this case every star product $\star_{\mathrm{red}}$ allows a trace density $\Omega = \Omega_0 + \cdots$ which is again unique up to a normalization in $\mathbb{R}[[\lambda]]$. In fact, there is even a canonical way to fix the normalization, see e.g. [19, 20, 23]. Thus in the symplectic case there is a preferred choice for μ yielding the complex conjugation as *-involution via Theorem 4.17.

We can now give another interpretation of Theorem 4.21. Two choices of the density μ (or equivalently, of Ω) yield *-involutions which are related by an *inner* automorphism. The question whether we can modify Ω to get the complex conjugation directly boils down to the question whether $I_\Omega = I_\mu$ from (84) is an inner automorphism or not. From Theorem 4.17 we known that

$$I_\Omega = I_\mu = \mathsf{id} + \sum_{r=1}^{\infty} \lambda^r I_r \tag{89}$$

with differential operators I_r depending on the choice of Ω. Any automorphism starting with the identity in zeroth order is necessarily of the form $I_\Omega = \exp(D_\Omega)$ with a *derivation*

$$D_\Omega = \sum_{r=1}^{\infty} \lambda^r D_\Omega^{(r)} \tag{90}$$

of the star product $\star_{\mathrm{red}}$, see e.g. [27, Prop. 6.2.7] or [10, Lem. 5]. The automorphism I_Ω changes by the inner automorphism $\mathrm{Ad}(\overline{\boldsymbol{\varrho}})$ when passing to Ω' according to Theorem 4.21. We arrive at the following result:

Proposition 4.25 (Modular class) *Let D_Ω be the derivation determined by Ω as above.*

i.) The first order term of D_Ω satisfies the classical infinitesimal KMS condition

$$\mathrm{i} \int_{M_{\mathrm{red}}} \{u, v\}_{\mathrm{red}} \Omega_0 + \int_{M_{\mathrm{red}}} D_\Omega^{(1)}(u) v \Omega_0 = 0 \tag{91}$$

for $u, v \in \mathcal{C}_0^\infty(M_{\mathrm{red}})$.

ii.) Denote the modular vector field of M_{red} with respect to Ω_0 by Δ_{Ω_0}. Then we have

$$D_\Omega^{(1)} = \mathrm{i}\Delta_{\Omega_0}. \tag{92}$$

iii.) For a different choice Ω' the difference $D_\Omega - D_{\Omega'}$ is an inner derivation of $\star_{\mathrm{red}}$. Hence the Hochschild cohomology class of D_Ω is independent of Ω.

Proof. The first part is the lowest non-vanishing order of (86). For the second, recall that the modular vector field Δ_{Ω_0} with respect to a positive density is defined by $\mathscr{L}_{X_u} \Omega_0 = \Delta_{\Omega_0}(u) \Omega_0$ where $u \in \mathcal{C}^\infty(M_{\mathrm{red}})$. Then (92) is clear from (91). Thus let Ω' be another choice and let $\boldsymbol{\varrho}$ be given as in Lemma 4.20. Then (88) gives

$$\exp(D_{\Omega'})(u) = I_{\Omega'}(u) = \boldsymbol{\varrho}^{-1} \star_{\mathrm{red}} I_\Omega(u) \star_{\mathrm{red}} \boldsymbol{\varrho} = \exp\left(-\operatorname{ad}_{\star_{\mathrm{red}}}(\mathrm{Log}(\varrho))\right)\left(\exp(D_\Omega)(u)\right),$$

where $\mathrm{Log}(\boldsymbol{\varrho}) = \log(\boldsymbol{\varrho}_0) + \cdots \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ is the $\star_{\mathrm{red}}$-logarithm of $\boldsymbol{\varrho}$. Indeed, this logarithm exists globally thanks to $\boldsymbol{\varrho}_0 > 0$ and it is unique up to constants in $2\pi\mathrm{i}\mathbb{Z}$, see [27, Sect. 6.3.1] for a detailed discussion of the logarithm with respect to star products. Since both derivations D_Ω and $-\operatorname{ad}_{\star_{\mathrm{red}}}(\mathrm{Log}(\boldsymbol{\varrho}))$ start in first order, their BCH series is well-defined. Thus

$$D_{\Omega'} = \mathrm{BCH}(-\operatorname{ad}_{\star_{\mathrm{red}}}(\mathrm{Log}(\boldsymbol{\varrho})), D_\Omega) = D_\Omega + \operatorname{ad}_{\star_{\mathrm{red}}}(w)$$

with some $w \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ since the commutators in the BCH series are all *inner* derivations.

□

Remark 4.26 On one hand, the proposition gives us a quantum analog of the modular class $[\Delta_{\Omega_0}]$ in the first Poisson cohomology as discussed in [28]. Indeed, the Hochschild cohomology class $[D_\Omega]$ of D_Ω is a deformation of $[\Delta_{\Omega_0}]$ in a very good sense and measures the analogous quantity, namely whether one can find a trace density. On the other hand, the proposition tells us that this *modular class* $[D_\Omega]$ of $\star_{\mathrm{red}}$ is precisely the obstruction for our construction of * to yield the complex conjugation by a clever choice of μ.

5 Construction of the inner product bimodule

Having constructed the reduced algebra $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$ out of the algebra $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ we want to relate their representation theories, i.e. their categories of modules, as well. From a physical point of view this is even crucial: ultimately, we need representations on some pre Hilbert space in order to establish the superposition principle, see e.g. [27, Chap.7] for a detailed discussion in the context of deformation quantization.

The usual idea is to use a bimodule and the tensor product to pass from modules of one algebra to modules of the other in a functorial way. Since we have constructed a bimodule structure on $\mathcal{C}^\infty(C)[[\lambda]]$ it is tempting to use this particular bimodule. While from a ring-theoretic point of view this is already interesting, we want to compare *-representations of the *-algebras on pre-Hilbert spaces and more generally on algebra-valued inner product modules. To this end, we want to add some more specific structure to the bimodule and make it an inner product bimodule with (ultimately) a completely positive inner product. The latter positivity will be discussed in Section 6, here we focus on the remaining properties of the inner product.

5.1 Algebra-valued inner products and *-representations of *-algebras

In this short subsection we collect some basic facts and definitions from [11]. One may recognize that all the notions are transferred from the theory of Hilbert modules over C^*-algebras to our more algebraic framework.

Let again R be an ordered ring and $\mathsf{C} = \mathsf{R}(\mathrm{i})$ as in Section 4.1 and consider a *-algebra $\mathcal{A}$ over C. Then an *$\mathcal{A}$-valued inner product* $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ on a right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ is a map $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{E}_{\mathcal{A}} \times \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{A}$ which is C-linear in the second argument and satisfies $\langle x, y \rangle_{\mathcal{A}} = (\langle y, x \rangle_{\mathcal{A}})^*$ as well as $\langle x, y \cdot a \rangle_{\mathcal{A}} = \langle x, y \rangle_{\mathcal{A}} a$. Moreover, we require non-degeneracy, i.e. $\langle x, y \rangle_{\mathcal{A}} = 0$ for all y implies $x = 0$. Here and in the following we always assume that every module over $\mathcal{A}$ carries a compatible C-module structure. If $\mathcal{E}_{\mathcal{A}}$ is equipped with such an inner product then $(\mathcal{E}_{\mathcal{A}}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$ is called an *inner product right $\mathcal{A}$-module* . Inner product left $\mathcal{A}$-modules are defined analogously, with the only difference that we require C-linearity and $\mathcal{A}$-linearity to the left in the *first* argument. For $\mathcal{A} = \mathsf{C}$ we get back the usual notions of an inner product module over the scalars C as in Section 4.1.

A map $A : \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{E}'_{\mathcal{A}}$ between inner product right $\mathcal{A}$-modules is called *adjointable* if there exists a map $A^* : \mathcal{E}'_{\mathcal{A}} \longrightarrow \mathcal{E}_{\mathcal{A}}$ with $\langle Ax, y \rangle^{\mathcal{E}'}_{\mathcal{A}} = \langle x, A^* y \rangle^{\mathcal{E}}_{\mathcal{A}}$ for all $x \in \mathcal{E}_{\mathcal{A}}$ and $y \in \mathcal{E}'_{\mathcal{A}}$. If such an A^* exists it is unique. It follows that A and A^* are right $\mathcal{A}$-linear and the adjointable maps form a C-submodule of $\mathsf{Hom}_{\mathsf{C}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$. Moreover, $A \mapsto A^*$ is C-antilinear and involutive. Finally, for another adjointable map $B : \mathcal{E}'_{\mathcal{A}} \longrightarrow \mathcal{E}''_{\mathcal{A}}$ also BA is adjointable with adjoint $(BA)^* = A^* B^*$. The adjointable maps are denoted by $\mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$.

A particular example of an adjointable map is obtained as follows: for $y \in \mathcal{E}'_{\mathcal{A}}$ and $x \in \mathcal{E}_{\mathcal{A}}$ we set

$$\Theta_{y,x}(z) = y \cdot \langle x, z \rangle^{\mathcal{E}}_{\mathcal{A}} \tag{93}$$

for all $z \in \mathcal{E}_{\mathcal{A}}$. This yields an adjointable operator $\Theta_{y,x} : \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{E}'_{\mathcal{A}}$ with adjoint $\Theta^*_{y,x} = \Theta_{x,y}$. The C-linear span of all these *rank one operators* are called the *finite rank operators*. They will be denoted by $\mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}'_{\mathcal{A}})$. As usual, we set $\mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}) = \mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}_{\mathcal{A}})$ and $\mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}) = \mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}}, \mathcal{E}_{\mathcal{A}})$. It follows that $\mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$ is a *-ideal in the unital *-algebra $\mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$.

If $\mathcal{B}$ is another *-algebra then a *-*representation* of $\mathcal{B}$ on an inner product right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ is a *-homomorphism $\pi : \mathcal{B} \longrightarrow \mathfrak{B}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$. This way, $\mathcal{E}_{\mathcal{A}}$ becomes a $(\mathcal{B}, \mathcal{A})$-bimodule and sometimes we simply write $b \cdot x = \pi(b)x$ if the map π is clear from the context. An *intertwiner* T between two such *-representations $(\mathcal{E}_{\mathcal{A}}, \pi)$ and $(\mathcal{E}'_{\mathcal{A}}, \pi')$ is a left $\mathcal{B}$-linear adjointable map $T : \mathcal{E}_{\mathcal{A}} \longrightarrow \mathcal{E}'_{\mathcal{A}}$, and hence in particular a $(\mathcal{B}, \mathcal{A})$-bimodule morphism. The category of *-representations of $\mathcal{B}$ on inner product right $\mathcal{A}$-modules is denoted by $^*\text{-}\mathsf{mod}_{\mathcal{A}}(\mathcal{B})$.

A *-representation ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in {}^*\text{-}\mathsf{mod}_{\mathcal{A}}(\mathcal{B})$ is called *strongly non-degenerate* if $\mathcal{B} \cdot {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. In the unital case this is equivalent to $\mathbb{1}_{\mathcal{B}} \cdot x = x$ for all $x \in {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. The subcategory of strongly non-degenerate *-representations is then denoted by $^*\text{-}\mathsf{Mod}_{\mathcal{A}}(\mathcal{B})$. Such *-representations will also be referred to as *inner product* $(\mathcal{B}, \mathcal{A})$*-bimodules*.

5.2 The definition of the inner product

The *-algebras in question will be the functions $\mathcal{C}^\infty(M)[[\lambda]]$ with $\star$ and the complex conjugation on one hand and $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ with $\star_{\mathrm{red}}$ and the complex conjugation on the other hand. Even though for $\star_{\mathrm{red}}$ we might also take the other *-involutions we restrict ourselves to the simplest case of the complex conjugation.

The first question is in which of the two *-algebras the inner product should take values. One option is ruled out by the following proposition:

Proposition 5.1 *Assume* $\operatorname{codim} C \geq 1$.

i.) For the classical $\mathcal{C}^\infty(M)$*-module structure of* $\mathcal{C}^\infty_0(C)$ *a* $\mathcal{C}^\infty(M)$*-valued inner product does not exist.*

ii.) For $\mathcal{C}^\infty_0(C)[[\lambda]]$ *there is no* $\mathcal{C}^\infty(M)[[\lambda]]$*-valued inner product with respect to* $\star$ *and the left module structure* $\bullet$.

Proof. Assume there is such an inner product. Let $\phi, \psi \in \mathcal{C}^\infty_0(C)$ with $\langle \phi, \psi \rangle \neq 0$ be given. Then there is a point $p \in M \setminus C$ with $\langle \phi, \psi \rangle (p) \neq 0$. Choose $f \in \mathcal{C}^\infty(M)$ with $f(p) \neq 0$ but f equal to zero in an open neighbourhood of C. Then we get a contradiction from $0 \neq f(p) \langle \phi, \psi \rangle (p) = \langle f \cdot \phi, \psi \rangle (p) = \langle \iota^* f \phi, \psi \rangle (p) = 0$ since $\iota^* f = 0$. This shows the first part. The second follows analogously, since $\boldsymbol{\iota}^*(f \star \operatorname{prol}(\phi)) = 0$ by the locality of $\star$ and Lemma 3.5. □

The other option of a $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued inner product will be more promising. Before giving the definition we have to specify the precise function space on C for the module: as we will need integrations, $\mathcal{C}^\infty(C)$ will be too large in general. On the other hand, $\mathcal{C}^\infty_0(C)$ will work but is too small for purposes of Morita theory in Section 6. Thus we shall use the following option: We define

$$
\mathcal{C}^\infty_{\mathrm{cf}}(C) = \left\{ \phi \in \mathcal{C}^\infty(C) \;\middle|\; \operatorname{supp}(\phi) \cap \pi^{-1}(K) \text{ is compact for all compact } K \subseteq M_{\mathrm{red}} \right\}, \tag{94}
$$

and call this subspace of $\mathcal{C}^\infty(C)$ the functions with *locally uniformly compact support in fiber directions.* Clearly $\mathcal{C}^\infty_0(C) \subseteq \mathcal{C}^\infty_{\mathrm{cf}}(C)$. If G is compact then $\mathcal{C}^\infty_{\mathrm{cf}}(C) = \mathcal{C}^\infty(C)$ while $\mathcal{C}^\infty_{\mathrm{cf}}(C) = \mathcal{C}^\infty_0(C)$ if M is compact. The importance of the space $\mathcal{C}^\infty_{\mathrm{cf}}(C)$ comes from the following simple observation, which becomes trivial for a compact group G.

Lemma 5.2 *i.) The subspace* $\mathcal{C}^\infty_{\mathrm{cf}}(C) \subseteq \mathcal{C}^\infty(C)$ *is an ideal, stable under all differential operators, the* G*-action, and complex conjugation.*

ii.) For $\phi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$ *the function*

$$
\int_G \mathsf{L}^*_{g^{-1}} \phi \, \mathrm{d}^{\mathrm{left}} g : c \;\mapsto\; \int_G \phi(\mathsf{L}_{g^{-1}}(c)) \, \mathrm{d}^{\mathrm{left}} g \tag{95}
$$

is a smooth and invariant function on C.

iii.) There exists a function $0 \leq \epsilon \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$ *with*

$$\int_G \mathsf{L}^*_{g^{-1}} \epsilon \, \mathrm{d}^{\mathrm{left}} g = 1. \tag{96}$$

Proof. The first part is trivial. For the second, let $U \subseteq C$ be an open pre-compact subset. Then $\phi(\mathsf{L}_{g^{-1}}(c)) = 0$ for $c \in U$ unless $\mathsf{L}_{g^{-1}}(c) \in \operatorname{supp} \phi$. On the other hand we know $\mathsf{L}^*_{g^{-1}}(c) \in \pi^{-1}(\pi(U^{\mathrm{cl}}))$ and $\operatorname{supp} \phi \cap \pi^{-1}(\pi(U^{\mathrm{cl}}))$ is compact thanks to $\phi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$. Thus $\phi(\mathsf{L}^*_{g^{-1}}(c)) = 0$ for $c \in U$ unless $g \in G_{U,\phi}$ where

$$G_{U,\phi} = \{ g \in G \mid \text{there exists a } c \in U^{\mathrm{cl}} \text{ with } \mathsf{L}_{g^{-1}}(c) \in \operatorname{supp} \phi \cap \pi^{-1}(\pi(U^{\mathrm{cl}})) \}.$$

Hence we conclude that for $c \in U$ we have

$$\int_G \phi(\mathsf{L}_{g^{-1}}(c)) \, \mathrm{d}^{\mathrm{left}} g = \int_{G_{U,\phi}} \phi(\mathsf{L}_{g^{-1}}(c)) \, \mathrm{d}^{\mathrm{left}} g.$$

Since $G_{U,\phi}$ is *compact* by the properness of the action we can deduce that (95) is well-defined and yields a smooth function on the open subset U by applying the usual "differentiation commutes with integration" techniques. But this implies smoothness everywhere. Clearly, the averaging integral yields an invariant function. For the third part, we use an atlas of local trivializations $\{U_\alpha, \Phi_\alpha\}$ of the (right) principal bundle. Moreover, let $0 \leq \chi_\alpha \in \mathcal{C}^\infty_0(M_{\mathrm{red}})$ be a locally finite partition of unity subordinate to this atlas. Finally, we choose $0 \leq \chi \in \mathcal{C}^\infty_0(G)$ with $\int_G \chi(g) \, \mathrm{d}^{\mathrm{left}} g = 1$. For $c \in C$ we define

$$\epsilon(c) = \sum\nolimits_\alpha (\chi_\alpha \otimes \chi) \circ \Phi_\alpha^{-1}(c).$$

It easily follows that $\epsilon \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$. Moreover, a simple computation shows that

$$\int_G (\chi_\alpha \otimes \chi) \circ \Phi_\alpha^{-1}(\mathsf{L}_{g^{-1}}(c)) \, \mathrm{d}^{\mathrm{left}} g = \chi_\alpha(\pi(c))$$

for all α. Thus ϵ satisfies (96). □

Corollary 5.3 *The* $\mathbb{C}[[\lambda]]$*-submodule* $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]] \subseteq \mathcal{C}^\infty(C)[[\lambda]]$ *is a* $\star_{\mathrm{red}}$*-submodule with respect to* $\bullet_{\mathrm{red}}$*.*

Proof. Since $\bullet_{\mathrm{red}}$ acts via differential operators in each order of λ, this is clear from Lemma 5.2, *i.)*. □

It will be this submodule on which the algebra-valued inner product can be defined.

Definition 5.4 (Algebra-valued inner product) *Let* $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$*. Then one defines their* $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$*-valued inner product* $\langle \phi, \psi \rangle_{\mathrm{red}}$ *pointwise by*

$$\langle \phi, \psi \rangle_{\mathrm{red}} (\pi(c)) = \int_G \left(\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \right) (\mathsf{L}_{g^{-1}}(c)) \, \mathrm{d}^{\mathrm{left}} g. \tag{97}$$

Lemma 5.5 *The inner product* $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ *is well-defined and* $\mathbb{C}[[\lambda]]$*-sesquilinear.*

Proof. The sesquilinearity is clear. Even though prol is a non-local operation it preserves the "support in C-directions". Using the locality of $\star$ and $\boldsymbol{\iota}^*$ we see that

$$\operatorname{supp} \left(\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \right) \subseteq \operatorname{supp} \phi \cap \operatorname{supp} \psi.$$

Thus the integrand of (97) is indeed in $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$. By Lemma 5.2 it follows that the right hand side of (97) is well-defined and yields an invariant smooth function on C. Hence it is of the form $\pi^* \langle \phi, \psi \rangle_{\mathrm{red}}$ with $\langle \phi, \psi \rangle_{\mathrm{red}} \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ as claimed. □

The next technical lemma shows alternative ways to compute $\langle \phi, \psi \rangle_{\mathrm{red}}$. Here again we rewrite the integral over G as an integral over a suitable compact subset.

Lemma 5.6 *Let $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$ and let $U \subseteq C$ be open and pre-compact.*

i.) $G_{U,\phi,\psi} = \{g \in G \mid$ *there exists a* $c \in U^{\mathrm{cl}}$ *with* $\mathsf{L}_{g^{-1}}(c) \in \operatorname{supp}\phi \cap \operatorname{supp}\psi \cap \pi^{-1}(\pi(U^{\mathrm{cl}}))\}$ *is a compact subset of* G*.*

ii.) One has

$$\pi^* \langle \phi, \psi \rangle_{\mathrm{red}} \Big|_U = \boldsymbol{\iota}^* \int_{G_{U,\phi,\psi}} \mathsf{L}^*_{g^{-1}} \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \mathrm{d}^{\mathrm{left}}\, g \Big|_U \tag{98}$$

$$= \boldsymbol{\iota}^* \int_{G_{U,\phi,\psi}} \overline{\mathrm{prol}(\mathsf{L}^*_{g^{-1}}\phi)} \star \mathrm{prol}(\mathsf{L}^*_{g^{-1}}\psi)\, \mathrm{d}^{\mathrm{left}}\, g \Big|_U. \tag{99}$$

Proof. By assumptions $\operatorname{supp}\phi \cap \operatorname{supp}\psi \cap \pi^{-1}(\pi(U^{\mathrm{cl}}))$ is compact. Then the properness of the action assures that $G_{U,\phi,\psi}$ is compact as well. A similar argument as in the proof of Lemma 5.2 shows that

$$\pi^* \langle \phi, \psi \rangle_{\mathrm{red}} \Big|_U = \int_{G_{U,\phi,\psi}} \mathsf{L}^*_{g^{-1}} \left(\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \right) \mathrm{d}^{\mathrm{left}}\, g \Big|_U.$$

Now by G-invariance of $\boldsymbol{\iota}^*$ we can exchange the action of $g \in G_{U,\phi,\psi}$ with $\boldsymbol{\iota}^*$. Moreover, since $\boldsymbol{\iota}^* = \iota^* \circ S$ is in each order of λ a differential operator followed by ι^*, see Lemma 3.5, the integration over the compact subset $G_{U,\phi,\psi}$ can be exchanged with $\boldsymbol{\iota}^*$. Thus (98) follows. Since $\star$ and prol are G-invariant, (99) follows as well. □

Lemma 5.7 *The inner product $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ is right $\star_{\mathrm{red}}$-linear, i.e. we have*

$$\langle \phi, \psi \bullet_{\mathrm{red}} u \rangle_{\mathrm{red}} = \langle \phi, \psi \rangle_{\mathrm{red}} \star_{\mathrm{red}} u \tag{100}$$

for all $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ and $u \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$.

Proof. First note that it suffices to consider $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$ and $u \in \mathcal{C}^\infty(M_{\mathrm{red}})$. We evaluate (100) on an open subset $U \subseteq C$ after pulling it back to C. In addition, we can assume U to be pre-compact. Then let $G_{U,\phi,\psi}$ be as in Lemma 5.6, *i.)*. For the integrand we have

$$\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi \bullet_{\mathrm{red}} u) \right) = \left(\overline{\mathrm{prol}(\phi)} \bullet \psi \right) \bullet_{\mathrm{red}} u = \boldsymbol{\iota}^* \left(\mathrm{prol} \left(\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \right) \star \mathrm{prol}(\pi^* u) \right)$$

by the bimodule properties as in Proposition 3.11. Since $\operatorname{supp}(\psi \bullet_{\mathrm{red}} u) \subseteq \operatorname{supp}\psi$, by Lemma 5.6 we get on the open subset U

$$\begin{aligned}
\pi^* \langle \phi, \psi \bullet_{\mathrm{red}} u \rangle_{\mathrm{red}} \Big|_U &= \boldsymbol{\iota}^* \int_{G_{U,\phi,\psi}} \mathsf{L}^*_{g^{-1}} \left(\mathrm{prol} \left(\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \right) \star \mathrm{prol}(\pi^* u) \right) \mathrm{d}^{\mathrm{left}}\, g \Big|_U \\
&= \boldsymbol{\iota}^* \left(\mathrm{prol} \left(\int_{G_{U,\phi,\psi}} \mathsf{L}^*_{g^{-1}} \left(\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \right) \mathrm{d}^{\mathrm{left}}\, g \right) \Big|_U \star \mathrm{prol}(\pi^* u) \Big|_U \right) \\
&= \boldsymbol{\iota}^* \left(\mathrm{prol} \left(\pi^* \langle \phi, \psi \rangle_{\mathrm{red}} \Big|_U \right) \star \mathrm{prol} \left(\pi^* u \Big|_U \right) \right) \\
&= \pi^* \left(\langle \phi, \psi \rangle_{\mathrm{red}} \star_{\mathrm{red}} u \right) \Big|_U,
\end{aligned}$$

where we have used that prol commutes with the integration thanks to the invariance. Moreover, we used the fact that we can restrict to open subsets on C: even though prol is non-local, the nice tubular neighbourhood shows that this is possible. □

Lemma 5.8 *Let $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$. Then we have*

$$\overline{\langle \phi, \psi \rangle_{\mathrm{red}}} = \langle \psi, \phi \rangle_{\mathrm{red}}. \tag{101}$$

Proof. Again, it will be sufficient to consider $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$. Let $U \subseteq C$ be open and pre-compact and let $G_{U,\phi,\psi}$ be as in Lemma 5.6, *i.)*. Then we compute

$$
\begin{aligned}
\overline{\langle \phi, \psi \rangle_{\mathrm{red}}}\Big|_U &= \overline{\boldsymbol{\iota}^* \int_{G_{U,\phi,\psi}} \mathsf{L}^*_{g^{-1}} \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \mathrm{d}^{\mathrm{left}}\, g \Big|_U} \\
&= \boldsymbol{\iota}^* \int_{G_{U,\phi,\psi}} \mathsf{L}^*_{g^{-1}} \left(\overline{\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi)} \right) \mathrm{d}^{\mathrm{left}}\, g \Big|_U \\
&= \pi^* \left\langle \psi, \phi \right\rangle_{\mathrm{red}} \Big|_U,
\end{aligned}
$$

since first the integration yields some invariant functions allowing to use (72) and, second, the star product $\star$ is Hermitian. $\square$

Lemma 5.9 *Let $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ then $\langle \phi, \phi \rangle_{\mathrm{red}} = 0$ iff $\phi = 0$. Moreover, the classical limit of the inner product is*

$$
\pi^* \left\langle \phi, \psi \right\rangle_{\mathrm{red}} = \int_G \mathsf{L}^*_{g^{-1}} (\overline{\phi} \psi)\, \mathrm{d}^{\mathrm{left}}\, g + \cdots . \tag{102}
$$

Proof. The classical limit (102) is clear. From this we also conclude the first statement by induction on the lowest non-vanishing order of ϕ. $\square$

We can now collect these results in the following theorem:

Theorem 5.10 *The inner product $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ turns $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ into an inner product right module over $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$.*

5.3 Further properties of $\langle \cdot, \cdot \rangle_{\mathrm{red}}$

Since also the action of $\mathcal{C}^\infty(M)[[\lambda]]$ via $\bullet$ on $\mathcal{C}^\infty(C)[[\lambda]]$ is by differential operators in each order, $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ is also a left $\star$-submodule of $\mathcal{C}^\infty(C)[[\lambda]]$ and hence a $(\star, \star_{\mathrm{red}})$-bimodule. The following proposition shows that the action is by adjointable operators and yields thus a *-representation:

Proposition 5.11 *The left module structure $\bullet$ is a *-representation of $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ on the inner product right $(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$-module $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$, i.e. we have for all $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ and $f \in \mathcal{C}^\infty(M)[[\lambda]]$*

$$
\left\langle \phi, f \bullet \psi \right\rangle_{\mathrm{red}} = \left\langle \overline{f} \bullet \phi, \psi \right\rangle_{\mathrm{red}} . \tag{103}
$$

Proof. Again, we consider $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$ and an open and pre-compact $U \subseteq C$ with $G_{U,\phi,\psi}$ as in Lemma 5.6, *i.)*. Since f acts by differential operators we have $\operatorname{supp}(f \bullet \psi) \subseteq \operatorname{supp} \psi$ as well as $\operatorname{supp}(\overline{f} \bullet \phi) \subseteq \operatorname{supp} \phi$. Analogously to the proof of Lemma 5.7 we first compute

$$
\begin{aligned}
\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(f \bullet \psi) \right) &= \overline{\mathrm{prol}(\phi)} \bullet (f \bullet \psi) \\
&= \left(\overline{\mathrm{prol}(\phi)} \star f \right) \bullet \psi \\
&= \left(\overline{\overline{f} \star \mathrm{prol}(\phi)} \right) \bullet \psi \\
&= \boldsymbol{\iota}^* \left(\left(\overline{\overline{f} \star \mathrm{prol}(\phi)} \right) \star \mathrm{prol}(\psi) \right) \\
&= \boldsymbol{\iota}^* \left(\overline{\overline{\mathrm{prol}(\psi)} \star \left(\overline{f} \star \mathrm{prol}(\phi) \right)} \right) .
\end{aligned}
$$

Using this we can take out the complex conjugation under the averaging integral since we have invariant functions thanks to (72). This gives

$$\pi^* \langle \phi, f \bullet \psi \rangle_{\mathrm{red}} \Big|_U = \overline{\int_{G_{U,\phi,\psi}} \mathsf{L}^*_{g^{-1}} \boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\psi)} \star (\overline{f} \star \mathrm{prol}(\phi)) \right) \mathrm{d}^{\mathrm{left}} g} \Big|_U,$$

where we again used Lemma 5.6. Now analogously one shows that

$$\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\psi)} \star \mathrm{prol}(\overline{f} \bullet \phi) \right) = \boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\psi)} \star (\overline{f} \star \mathrm{prol}(\phi)) \right),$$

using the definition of the left module structure. Putting these together shows (103). □

The last feature of $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ we want to discuss is the G-invariance. In fact, the G-action of $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ turns out to be unitary with respect to $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ up to the modular function:

Proposition 5.12 *Let $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ and $g \in G$ then*

$$\left\langle \mathsf{L}^*_{g^{-1}} \phi, \mathsf{L}^*_{g^{-1}} \psi \right\rangle_{\mathrm{red}} = \Delta(g) \langle \phi, \psi \rangle_{\mathrm{red}}. \tag{104}$$

Proof. Using the G-equivariance of all operators involved in the definition of $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ the computation is analogous to the one for the classical limit (102) which is clear. □

Thus we obtain a unitary (left) representation of G with respect to $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ if we set

$$U(g)\phi = \frac{1}{\sqrt{\Delta(g)}} \mathsf{L}^*_{g^{-1}} \phi. \tag{105}$$

The infinitesimal version of this action is given by the left multiplication with the components of the momentum map as in (47): here we again see that $\kappa = \frac{1}{2}$ is the good choice to get this unitarity. From Proposition 3.7, *iv.)* we see that the left module structure is covariant:

Corollary 5.13 *The *-representation $\bullet$ if $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ is G-covariant with respect to the unitary representation U of G as in* (105)*, i.e.*

$$U(g)(f \bullet \phi) = (\mathsf{L}^*_{g^{-1}} f) \bullet (U(g)\phi) \tag{106}$$

for $f \in \mathcal{C}^\infty(M)[[\lambda]]$, $g \in G$, and $\phi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$.

6 A strong Morita equivalence bimodule

In this section we establish that the bimodule structure and inner product $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ actually gives a strong Morita equivalence bimodule between $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ and the finite rank operators on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$.

6.1 Strong Morita equivalence of *-algebras

There are many approaches to Morita theory of *-algebras, see e.g. [1, 11] for a detailed discussion and further references. We recall the basic notions: If ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in {}^*\text{-}\mathsf{mod}_{\mathcal{A}}(\mathcal{B})$ and ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \in {}^*\text{-}\mathsf{mod}_{\mathcal{B}}(\mathcal{C})$ are *-representations on inner product modules then on their algebraic tensor product one defines an $\mathcal{A}$-valued inner product by

$$\langle \phi \otimes x, \psi \otimes y \rangle^{\mathcal{F} \otimes \mathcal{E}}_{\mathcal{A}} = \left\langle x, \langle \phi, \psi \rangle^{\mathcal{F}}_{\mathcal{B}} \cdot y \right\rangle^{\mathcal{E}}_{\mathcal{A}} \tag{107}$$

and extends this by C-sesquilinearity to $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E}$. It follows that this is again an inner product except, however, it might be degenerate. Thus one considers the quotient

$$ {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} = {}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \,/\, ({}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \otimes_{\mathcal{B}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}})^{\perp}, \tag{108} $$

and obtains a *-representation of $\mathcal{C}$ on an inner product right $\mathcal{A}$-module, called the *internal tensor product* ${}_{\mathcal{C}}\mathcal{F}_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$. It turns out that $\widehat{\otimes}$ gives a functor

$$ \widehat{\otimes} : {}^*\text{-}\mathsf{mod}_{\mathcal{B}}(\mathcal{C}) \times {}^*\text{-}\mathsf{mod}_{\mathcal{A}}(\mathcal{B}) \longrightarrow {}^*\text{-}\mathsf{mod}_{\mathcal{A}}(\mathcal{C}), \tag{109} $$

which preserves strongly non-degenerate *-representations. This allows for the following definition: a *-representation ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in {}^*\text{-}\mathsf{Mod}_{\mathcal{A}}(\mathcal{B})$ is called a **-equivalence bimodule* if there exists another *-representation ${}_{\mathcal{A}}\widetilde{\mathcal{E}}_{\mathcal{B}} \in {}^*\text{-}\mathsf{mod}_{\mathcal{B}}(\mathcal{A})$ such that

$$ {}_{\mathcal{A}}\widetilde{\mathcal{E}}_{\mathcal{B}} \mathbin{\widehat{\otimes}_{\mathcal{B}}} {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \cong \mathcal{A} \quad \text{and} \quad {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \mathbin{\widehat{\otimes}_{\mathcal{A}}} {}_{\mathcal{A}}\widetilde{\mathcal{E}}_{\mathcal{B}} \cong \mathcal{B}, \tag{110} $$

where isomorphism is understood as unitary intertwiner of *-representations with the algebras being equipped with the canonical inner product, i.e. $\langle a, a' \rangle_{\mathcal{A}} = a^* a'$. In order to have $\mathcal{A} \in {}^*\text{-}\mathsf{mod}_{\mathcal{A}}(\mathcal{A})$ we have to restrict ourselves to *non-degenerate* *-algebras, i.e. $ab = 0$ for all b implies $a = 0$. Moreover, one has to require the *-algebras to be *idempotent*, i.e. $\mathcal{A} \cdot \mathcal{A} = \mathcal{A}$. Every unital *-algebra is both, non-degenerate and idempotent. For this class of *-algebras it can be checked that the existence of a *-equivalence bimodule defines indeed an equivalence relation, called **-Morita equivalence*.

The following characterization of *-equivalence bimodules will be very useful for us: ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \in {}^*\text{-}\mathsf{mod}_{\mathcal{A}}(\mathcal{B})$ is a *-equivalence bimodule iff the following holds: First, $\mathcal{B} \cdot \mathcal{E} = \mathcal{E} = \mathcal{E} \cdot \mathcal{A}$, i.e. both module structures are strongly non-degenerate. Second, there is a $\mathcal{B}$-valued inner product ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$ with ${}_{\mathcal{B}}\langle x \cdot a, y \rangle = {}_{\mathcal{B}}\langle x, y \cdot a^* \rangle$ for all $x, y \in \mathcal{E}$ and $a \in \mathcal{A}$. Third, both inner products are *full*, i.e. $\langle \mathcal{E}, \mathcal{E} \rangle_{\mathcal{A}} = \mathcal{A}$ and ${}_{\mathcal{B}}\langle \mathcal{E}, \mathcal{E} \rangle = \mathcal{B}$. Last, the inner products are compatible, i.e. ${}_{\mathcal{B}}\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_{\mathcal{A}}$ for all $x, y, z \in \mathcal{E}$. In this case, the "inverse" bimodule ${}_{\mathcal{A}}\widetilde{\mathcal{E}}_{\mathcal{B}}$ can be chosen to be the complex conjugate bimodule ${}_{\mathcal{A}}\overline{\mathcal{E}}_{\mathcal{B}}$, defined in an obvious way. Moreover, $\mathcal{B}$ turns out to be *-isomorphic to the finite rank operators $\mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$ via the left module structure and ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$ is just $\Theta_{\cdot,\cdot}$ as in (93) under this identification, see [11] for a detailed discussion.

In fact, we only have to find a strongly non-degenerate right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ with a full inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ then $\mathcal{B} = \mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$ is *-Morita equivalent to $\mathcal{A}$ via ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ where the $\mathfrak{F}_{\mathcal{A}}(\mathcal{E}_{\mathcal{A}})$-valued inner product is $\Theta_{\cdot,\cdot}$. All *-Morita equivalent *-algebras to $\mathcal{A}$ arise this way up to *-isomorphism.

Let us finally recall the notion of strong Morita equivalence. First recall that $a \in \mathcal{A}$ is called *positive* if $\omega(a) \geq 0$ for all positive linear functionals $\omega : \mathcal{A} \longrightarrow \mathsf{C}$. This allows to define an inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ on a right $\mathcal{A}$-module $\mathcal{E}_{\mathcal{A}}$ to be *completely positive* if the matrix $(\langle x_i, x_j \rangle_{\mathcal{A}}) \in M_n(\mathcal{A})$ is positive for all $x_1, \ldots, x_n \in \mathcal{E}_{\mathcal{A}}$ and $n \in \mathbb{N}$. This gives a refined notion of inner product modules: an inner product right $\mathcal{A}$-module is called *pre Hilbert right* $\mathcal{A}$*-module* if the inner product is completely positive. The category of *-representations of $\mathcal{B}$ on pre Hilbert right $\mathcal{A}$-modules is denoted by ${}^*\text{-}\mathsf{rep}_{\mathcal{A}}(\mathcal{B})$ and the sub-category of strongly non-degenerate ones by ${}^*\text{-}\mathsf{Rep}_{\mathcal{A}}(\mathcal{B})$. It can be shown that the internal tensor product $\widehat{\otimes}$ preserves complete positivity of the inner products. Moreover, the canonical inner product on $\mathcal{A}$ is easily shown to be completely positive. This allows to define a *-equivalence bimodule to be a *strong equivalence bimodule* if both inner products are completely positive. This way one arrives at the notion of *strong Morita equivalence*.

One of the most important consequences of *-Morita equivalence and strong Morita equivalence is that the *-representation theories ${}^*\text{-}\mathsf{Mod}_{\mathcal{D}}(\cdot)$ and ${}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\cdot)$ of equivalent *-algebras are equivalent, even for arbitrary "coefficient *-algebra" $\mathcal{D}$. In fact, every equivalence bimodule ${}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}}$ provides an equivalence of categories by the internal tensor product

$$ {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \mathbin{\widehat{\otimes}_{\mathcal{A}}} : {}^*\text{-}\mathsf{Mod}_{\mathcal{D}}(\mathcal{A}) \longrightarrow {}^*\text{-}\mathsf{Mod}_{\mathcal{D}}(\mathcal{B}) \quad \text{and} \quad {}_{\mathcal{B}}\mathcal{E}_{\mathcal{A}} \mathbin{\widehat{\otimes}_{\mathcal{A}}} : {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{A}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{B}), \tag{111} $$

in the case of strong Morita equivalence, respectively. We refer to [11] for further details on strong Morita equivalence of *-algebras.

6.2 Fullness and the finite rank operators

We want to investigate the inner product bimodule $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ from the point of view of Morita theory. The first result is the fullness based on the following lemma:

Lemma 6.1 *There exists a function* $\boldsymbol{e} \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ *such that*

$$\langle \boldsymbol{e}, \boldsymbol{e} \rangle_{\mathrm{red}} = 1. \tag{112}$$

Proof. We consider the function $\epsilon \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$ from Lemma 5.2, *iii.)*. Since each term $\Phi^*_\alpha(\chi_\alpha \otimes \chi) \geq 0$ is already non-negative, we obtain

$$\pi^* \langle \epsilon, \epsilon \rangle_{\mathrm{red}} = \sum_{\alpha,\beta} \int_G \mathsf{L}^*_{g^{-1}} \left(\overline{\Phi^*_\alpha(\chi_\alpha \otimes \chi)} \Phi^*_\beta(\chi_\beta \otimes \chi) \right) \mathrm{d}^{\mathrm{left}} g + \cdots$$

by (102). Now all integrands are non-negative and since $\int_G \mathsf{L}^*_{g^{-1}} \epsilon \, \mathrm{d}^{\mathrm{left}} g = 1$ we see that already the diagonal terms give a strictly positive contribution. It follows that $\langle \epsilon, \epsilon \rangle_{\mathrm{red}} = u_0 + \cdots \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ with $u_0 > 0$. Since in addition $\overline{\langle \epsilon, \epsilon \rangle_{\mathrm{red}}} = \langle \epsilon, \epsilon \rangle_{\mathrm{red}}$ is Hermitian, we can take a (Hermitian) square root with respect to $\star_{\mathrm{red}}$ of the form $\sqrt[\star_{\mathrm{red}}]{\langle \epsilon, \epsilon \rangle} = \sqrt{u_0} + \cdots$. Clearly, this is invertible hence $\boldsymbol{e} = \epsilon \star_{\mathrm{red}} \frac{1}{\sqrt[\star_{\mathrm{red}}]{\langle \epsilon, \epsilon \rangle}}$ will do the job. □

Note that if G is compact and $\mathrm{d}^{\mathrm{left}} g$ normalized to volume 1 then $\boldsymbol{e} = 1 \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$ would be a canonical choice.

Proposition 6.2 (*-Equivalence bimodule) *For* $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ *we have:*

i.) The inner product $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ *is full.*

ii.) The inner product right $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$*-module* $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ *becomes a* **-equivalence bimodule for the finite rank operators* $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]])$ *acting from the left as usual with* $\Theta_{\cdot,\cdot}$ *as inner product.*

iii.) As left $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]])$*-module,* $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ *is cyclic with cyclic vector* $\boldsymbol{e}$*. Moreover, for all* $\phi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$

$$\phi = \Theta_{\phi,\boldsymbol{e}}(\boldsymbol{e}). \tag{113}$$

iv.) The pair $(\boldsymbol{e}, \boldsymbol{e})$ *constitutes a Hermitian dual basis hence* $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ *is finitely generated (by* $\boldsymbol{e}$*) and projective over* $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]])$.

v.) The inner product $\Theta_{\cdot,\cdot}$ *is completely positive.*

Proof. Since $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ is unital, (112) is sufficient to conclude fullness. Then the second part is clear by the general structure of *-equivalence bimodules. Now (113) is just a computation using (112). This means that $\boldsymbol{e}$ is cyclic for the action of $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]])$. Even more, since the inner product $\Theta_{\phi,\boldsymbol{e}}$ is $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]])$-linear to the left in the first argument ϕ, we have a dual basis $(\boldsymbol{e}, \Theta_{\cdot,\boldsymbol{e}})$ for the left $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]])$-module $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$. Since the linear form $\Theta_{\cdot,\boldsymbol{e}}$ is obviously an inner product by some vector, namely $\boldsymbol{e}$, this is even a Hermitian dual basis, see [11]. The existence of such a Hermitian dual basis is true in general since the other algebra is unital. The remarkable point is that we only need one vector $\boldsymbol{e}$. For the last part, let $\phi_1, \ldots, \phi_n \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ be given and consider the matrix $\Phi = (\Theta_{\phi_i,\phi_j}) \in M_n(\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]))$. From (112) it immediately follows that $\Phi = \Psi^* \Psi$ where Ψ is the matrix with $\Theta_{\boldsymbol{e},\phi_i}$ in the first row and zeros elsewhere. Thus Φ is clearly positive proving the last part. □

Remark 6.3 Assume that G is not finite, which we always can assume in the context of phase space reduction. Then the finite rank operators do *not* have a unit. Otherwise, the module would be also finitely generated and projective as right $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-module. This is clearly not the case as the finitely projective modules over star product algebras are known to be deformations of sections of vector bundles over the base manifold. Thus we have a first non-trivial example of a *-equivalence bimodule for star product algebras going beyond the unital case studied in [10].

Remark 6.4 Note also that the *-algebra $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ does *not* act via finite rank operators on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$. The reason is that the finite rank operators are *non-local* as they involve the integration along the fibers in the definition of $\langle \cdot, \cdot \rangle_{\mathrm{red}}$. However, we know that $\phi \mapsto f \bullet \phi$ is a formal series of differential and hence local operators. Of course, we can not expect $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ and $(\mathcal{C}^\infty_{\mathrm{cf}}(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$ to be *-equivalent as in this case the classical limit of this bimodule would be still an equivalence bimodule and thus M is necessarily diffeomorphic to M_{red}, see [9, Cor. 7.8].

In the rest of this subsection we discuss the classical limit of the *-equivalence bimodule and the finite rank operators: we consider $\mathcal{C}^\infty_{\mathrm{cf}}(C)$ as right $\mathcal{C}^\infty(M_{\mathrm{red}})$-module with inner product

$$\langle \phi, \psi \rangle^{\mathrm{cl}}_{\mathrm{red}} = \int_G \mathsf{L}^*_{g^{-1}}(\overline{\phi}\psi)\, \mathrm{d}^{\mathrm{left}} g. \tag{114}$$

Since we do not rely on phase space reduction, the Lie group G needs not to be connected in the following theorem:

Theorem 6.5 *Let $C \circlearrowleft G \longrightarrow M_{\mathrm{red}}$ be a principal bundle.*

i.) The inner product $\langle \cdot, \cdot \rangle^{\mathrm{cl}}_{\mathrm{red}}$ is full, non-degenerate and completely positive.

ii.) The pre Hilbert right $\mathcal{C}^\infty(M_{\mathrm{red}})$-module $\mathcal{C}^\infty_{\mathrm{cf}}(C)$ becomes a strong Morita equivalence bimodule for the finite rank operators $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C))$ acting from the left with the canonical inner product $\Theta_{\cdot,\cdot}$.

Proof. Clearly, $\langle \cdot, \cdot \rangle^{\mathrm{cl}}_{\mathrm{red}}$ is a well-defined $\mathcal{C}^\infty(M_{\mathrm{red}})$-valued inner product which is non-degenerate. Analogously to the construction of $\boldsymbol{e}$ we find a function $e \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$ with $\langle e, e \rangle^{\mathrm{cl}}_{\mathrm{red}} = 1$ showing fullness. For $\phi_1, \ldots, \phi_n \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$ the matrix $(\langle \phi_i, \phi_j \rangle^{\mathrm{cl}}_{\mathrm{red}}) \in M_n(\mathcal{C}^\infty(M_{\mathrm{red}}))$ is pointwise positive. But this is precisely the characterization of positive elements in $M_n(\mathcal{C}^\infty(M_{\mathrm{red}}))$ resulting from the general algebraic definition, see also [9, App. B]. Finally, the same argument as in the proof of Proposition 6.2 using e instead of $\boldsymbol{e}$ shows the complete positivity for $\Theta_{\cdot,\cdot}$ also in this case. Thus the second statement follows. □

In order to get a more geometric description of the finite rank operators we consider the "extended" principal bundle

$$\pi_e : C_e = C \times_{M_{\mathrm{red}}} C \longrightarrow M_{\mathrm{red}}, \tag{115}$$

which is indeed a $G \times G$ principal bundle over M_{red}. Denote by $\mathrm{pr}_1, \mathrm{pr}_2 : C \times_{M_{\mathrm{red}}} C \longrightarrow C$ the projections onto the first and second factor of the fiber product, respectively. We can define $\mathcal{C}^\infty_{\mathrm{cf}}(C_e)$ analogously to the case of $\mathcal{C}^\infty_{\mathrm{cf}}(C)$. On this space we define a "matrix multiplication" by

$$(\Psi * \Xi)(c, c') = \int_G \Psi(c, \mathsf{L}_{g^{-1}}(\tilde{c}))\Xi(\mathsf{L}_{g^{-1}}(\tilde{c}), c')\, \mathrm{d}^{\mathrm{left}} g \tag{116}$$

for $(c, c') \in C_e$ and $\Phi, \Xi \in \mathcal{C}^\infty_{\mathrm{cf}}(C_e)$ where $\tilde{c}$ is an arbitrary point in the same fiber as c and c'. Moreover, one defines an action of $\mathcal{C}^\infty_{\mathrm{cf}}(C_e)$ on $\mathcal{C}^\infty_{\mathrm{cf}}(C)$ by

$$(\Psi \cdot \phi)(c) = \int_G \Psi(c, \mathsf{L}_{g^{-1}}(\tilde{c}))\phi(\mathsf{L}_{g^{-1}}(\tilde{c}))\, \mathrm{d}^{\mathrm{left}} g, \tag{117}$$

where now $\Psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C_e)$, $\phi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$, and $\tilde{c}$ is again an arbitrary point in the same fiber as c. These definitions turn out to make sense and have the following properties:

Theorem 6.6 *Let $\Psi, \Xi \in \mathcal{C}^\infty_{\mathrm{cf}}(C_e)$ and $\phi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$.*

i.) The product (116) *is well-defined, independent on the choice of $\tilde{c}$ and yields a smooth function $\Psi * \Xi \in \mathcal{C}^\infty_{\mathrm{cf}}(C_e)$ with*

$$\mathrm{supp}(\Psi * \Xi) \subseteq \mathrm{pr}_1(\mathrm{supp}\, \Psi) \times_{M_{\mathrm{red}}} \mathrm{pr}_2(\mathrm{supp}\, \Xi). \tag{118}$$

ii.) *Together with the* **-involution defined for* $\Psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C_e)$ *by*

$$\Psi^*(c, c') = \overline{\Psi(c', c)} \tag{119}$$

the product $*$ *turns* $\mathcal{C}^\infty_{\mathrm{cf}}(C_e)$ *into a* **-algebra over* $\mathbb{C}$.

iii.) *The definition* (117) *is independent on* $\tilde{c}$ *and yields a smooth function* $\Psi \cdot \phi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$. *This way,* $\mathcal{C}^\infty_{\mathrm{cf}}(C)$ *becomes a left* $\mathcal{C}^\infty_{\mathrm{cf}}(C_e)$*-module. The map* $\Psi \mapsto (\phi \mapsto \Psi \cdot \phi)$ *is injective.*

iv.) *One obtains a pre Hilbert* $(\mathcal{C}^\infty_{\mathrm{cf}}(C_e), \mathcal{C}^\infty(M_{\mathrm{red}}))$*-bimodule with respect to the* $\mathcal{C}^\infty(M_{\mathrm{red}})$*-valued inner product* $\langle \cdot, \cdot \rangle^{\mathrm{cl}}_{\mathrm{red}}$.

v.) *The linear map determined by*

$$\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)) \ni \Theta_{\phi,\psi} \;\mapsto\; \phi \otimes \overline{\psi} = ((c, c') \mapsto \phi(c)\overline{\psi(c')}) \in \mathcal{C}^\infty_{\mathrm{cf}}(C_e) \tag{120}$$

yields an injective **-algebra homomorphism such that the left module multiplication* (117) *of the images under* (120) *coincides with the canonical action of the finite rank operators.*

Proof. For the pointwise existence we note that those $g \in G$ with $\mathsf{L}_{g^{-1}}(\tilde{c}) \in \operatorname{supp} \Psi(c, \cdot)$ or $\mathsf{L}_{g^{-1}}(\tilde{c}) \in \operatorname{supp} \Xi(\cdot, c')$, respectively, are compact by the properness of the action and the assumption $\Psi, \Xi \in \mathcal{C}^\infty_{\mathrm{cf}}(C_e)$. Thus the integral (116) only uses the $g \in G$ in the intersection of these two compact subsets. This shows convergence of the integral. The independence on $\tilde{c}$ follows from the left invariance of $\mathrm{d}^{\mathrm{left}} g$ at once. To show smoothness of (116) we need a locally uniform compact domain of integration. Thus let $U_e \subseteq C_e$ be open and pre-compact and assume that $\pi_e(U_e) \subseteq M_{\mathrm{red}}$ allows for a trivialization of C over this open subset. Thus we can choose a smooth local section $\tilde{c} : \pi_e(U_e) \longrightarrow \pi^{-1}(\pi_e(U_e)) \subseteq C$. By assumption, the subsets $\operatorname{supp} \Psi \cap \pi_e^{-1}(\pi_e(U_e^{\mathrm{cl}}))$ and $\operatorname{supp} \Xi \cap \pi_e^{-1}(\pi_e(U_e^{\mathrm{cl}}))$ are compact in C_e. Thus also $K_\Psi = \mathrm{pr}_2(\operatorname{supp} \Psi \cap \pi_e^{-1}(\pi_e(U_e^{\mathrm{cl}})))$ and $K_\Xi = \mathrm{pr}_1(\operatorname{supp} \Xi \cap \pi_e^{-1}(\pi_e(U_e^{\mathrm{cl}})))$ are compact subsets of C projecting into $\pi_e(U_e^{\mathrm{cl}})$. Then by the properness of the action we see that $G_\Psi = \{g \in G \mid \text{there exists a } c \in \mathrm{pr}_1(U^{\mathrm{cl}}) \text{ with } \mathsf{L}_{g^{-1}}(\tilde{c}(\pi(c))) \in K_\Psi\}$ as well as $G_\Xi = \{g \in G \mid \text{there exists a } c' \in \mathrm{pr}_2(U^{\mathrm{cl}}) \text{ with } \mathsf{L}_{g^{-1}}(\tilde{c}(\pi(c'))) \in K_\Xi\}$ are compact subsets of G. By construction, the integration only needs the $g \in G_\Psi \cap G_\Xi$ for all $(c, c') \in U_e$. Thus on U_e, we have a uniform compact domain of integration. Then the smoothness follows easily. The statement (118) is clear from which we also deduce that $\Psi * \Xi \in \mathcal{C}^\infty_{\mathrm{cf}}(C_e)$, showing the first part. For the second part we note that (119) is clearly an involutive and anti-linear endomorphism of $\mathcal{C}^\infty_{\mathrm{cf}}(C_e)$. A simple computation shows that this indeed gives an anti-automorphism of $*$. It remains to show the associativity of the product $*$ which is an easy consequence of Fubini's theorem as locally we only have to integrate over compact subsets of G. For the third part, one proceeds analogously to show $\Psi \cdot \phi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)$, independently on the choice of $\tilde{c}$. The module property is again an application of Fubini's theorem. The injectivity is clear. For the fourth part we have to show that (117) is a *-representation. Thus we compute

$$\begin{aligned}
\langle \phi, \Xi \cdot \psi \rangle^{\mathrm{cl}}_{\mathrm{red}} (\pi(c)) &= \int_G \overline{\phi(\mathsf{L}_{g^{-1}}(c))} (\Xi \cdot \psi)(\mathsf{L}_{g^{-1}}(c)) \, \mathrm{d}^{\mathrm{left}} g \\
&= \int_G \overline{\phi(\mathsf{L}_{g^{-1}}(c))} \int_G \Xi(\mathsf{L}_{g^{-1}}(c), \mathsf{L}_{h^{-1}}(\tilde{c})) \psi(\mathsf{L}_{h^{-1}}(\tilde{c})) \, \mathrm{d}^{\mathrm{left}} h \, \mathrm{d}^{\mathrm{left}} g \\
&= \int_G \overline{\int_G \Xi^*(\mathsf{L}_{h^{-1}}(\tilde{c}), \mathsf{L}_{g^{-1}}(c)) \phi(\mathsf{L}_{g^{-1}}(c)) \, \mathrm{d}^{\mathrm{left}} g} \, \psi(\mathsf{L}_{h^{-1}}(\tilde{c}) \, \mathrm{d}^{\mathrm{left}} h \\
&= \langle \Xi^* \cdot \phi, \psi \rangle^{\mathrm{cl}}_{\mathrm{red}} (\pi(\tilde{c})),
\end{aligned}$$

using once again Fubini's theorem. Since $\pi(\tilde{c}) = \pi(c)$ the fourth part follows. For the last part we first note that clearly $\phi \otimes \overline{\psi} \in \mathcal{C}^\infty_{\mathrm{cf}}(C_e)$. We compute

$$((\phi \otimes \overline{\psi}) \cdot \chi)(c) = \int_G \phi(c) \overline{\psi(\mathsf{L}_{g^{-1}}(\tilde{c}))} \chi(\mathsf{L}_{g^{-1}}(\tilde{c})) \, \mathrm{d}^{\mathrm{left}} g = \phi(c) \, \langle \psi, \chi \rangle^{\mathrm{cl}}_{\mathrm{red}} (\pi(c)) = (\Theta_{\phi,\psi} \chi)(c).$$

This shows that under (120) the usual action of finite rank operators is turned into (117). By the injectivity statement in the third part, (120) is necessarily injective and a *-homomorphism. □

Remark 6.7 The *-algebra $\mathcal{C}^\infty_{\mathrm{cf}}(C_e)$ is typically strictly larger than the image of $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C))$ under the embedding (120). Nevertheless, with respect to a suitable locally convex topology, the finite rank operators are dense in $\mathcal{C}^\infty_{\mathrm{cf}}(C_e)$. Morally, $\mathcal{C}^\infty_{\mathrm{cf}}(C_e)$ corresponds to "Hilbert-Schmidt"-like operators. Note that the product $*$ can be viewed as a "matrix-multiplication" of matrices with components labeled by the continuous index $g \in G$. Similarly, the left module structure (117) is the application of a matrix to a vector whose components are labelled by the continuous index $g \in G$. Finally, the *-involution is the usual "matrix-adjoint".

Remark 6.8 Geometrically, the bundle $C_e = C \times_{M_{\mathrm{red}}} C \longrightarrow M_{\mathrm{red}}$ is diffeomorphic to $C \times G$ since for $(c, c') \in C_e$ there exists a unique $g \in G$ with $c' = \mathsf{L}_{g^{-1}}(c)$. However, this diffeomorphism will destroy the simple form of the matrix-multiplication formulas (116) and (117). Nevertheless, rewriting things this way, one recognizes the usual crossed product construction, here in its "smooth" version: the smooth functions on the reduced space M_{red} are *-Morita equivalent to the crossed product of the functions on C with the group G. Of course, since G is non-compact, some care has to be taken and the above function space provides a good notion for the crossed product in the smooth situation, see [25] for the original version of this statement in the C^*-algebraic category.

6.3 Complete positivity

Before showing the complete positivity of $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ we recall some facts on deformation quantization of principal bundles $C \circlearrowright G \longrightarrow M_{\mathrm{red}}$ from [8]: it can be shown that $\mathcal{C}^\infty(C)[[\lambda]]$ can always be equipped with a right module structure $\bullet_{\mathrm{red}}$ with respect to a given star product $\star_{\mathrm{red}}$ on M_{red}. Moreover, $\bullet_{\mathrm{red}}$ is *unique* up to equivalence, i.e. up to a module isomorphism of the form $\mathsf{id} + \sum_{r=1}^\infty \lambda^r T_r$ with $T_r \in \mathrm{DiffOp}(C)$. Moreover, it is known that the module endomorphisms of such a deformation inside the differential operators $\mathrm{DiffOp}(C)[[\lambda]]$ are obtained from a deformation of the *vertical* differential operators $\mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$, now equipped with a new, deformed composition law $\star'$ and a deformed action $\bullet'$ on $\mathcal{C}^\infty(C)[[\lambda]]$. Again, $\star'$ and $\bullet'$ are uniquely determined by $\star_{\mathrm{red}}$ up to equivalence. We shall use these results, in particular the uniqueness statements, later on.

Remark 6.9 (Positive algebra elements) We have to make an additional requirement on the positive linear functionals $\omega = \sum_{r=0}^\infty \lambda^r \omega_r : \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]]$ for the following. While the algebraic definition (57) allows for ω_r in the full algebraic dual of $\mathcal{C}^\infty(M_{\mathrm{red}})$ we have to restrict to *distributions*, i.e. continuous linear functionals with respect to the canonical Fréchet topology of $\mathcal{C}^\infty(M_{\mathrm{red}})$. It is easy to construct (algebraically) positive linear functionals where the higher orders are not of this form. However, this restriction seems to be reasonable as long as we work with smooth functions. Potentially, this will result in *more* positive algebra elements.

We start now with the local situation: we consider an open and small enough subset $U \subseteq M_{\mathrm{red}}$ and assume that $\pi^{-1}(U) \cong U \times G$ is trivial. Following the principal bundle tradition, the group acts from the *right* by right multiplications denoted by $\mathsf{r}_g : U \times G \longrightarrow U \times G$. The corresponding left action is therefor given by $\mathsf{L}_g = \mathsf{r}_{g^{-1}}$, and *not* by the left multiplication l_g.

The star product $\star_{\mathrm{red}}$ extends canonically to $\mathcal{C}^\infty(U \times G)[[\lambda]]$ yielding a star product, still denoted by $\star_{\mathrm{red}}$, for the Poisson structure on $U \times G$ which is the flat horizontal lift of the one on U. This way,

$$\pi^* : (\mathcal{C}^\infty(U)[[\lambda]], \star_{\mathrm{red}}) \longrightarrow (\mathcal{C}^\infty(U \times G)[[\lambda]], \star_{\mathrm{red}}) \tag{121}$$

is a *-algebra homomorphism. Thus we also obtain a canonical right $\star_{\mathrm{red}}$-module structure $\bullet_{\mathrm{can}}$ on $\mathcal{C}^\infty(U \times G)[[\lambda]]$ using (121). For this particular right module structure we can define a very simple inner product. Indeed, for $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(U \times G)[[\lambda]]$ we set

$$\pi^* \langle \phi, \psi \rangle_{\mathrm{can}} = \int_G \mathsf{L}^*_{g^{-1}}(\overline{\phi} \star_{\mathrm{red}} \psi) \, \mathrm{d}^{\mathrm{left}} g = \int_G \mathsf{r}^*_g(\overline{\phi} \star_{\mathrm{red}} \psi) \, \mathrm{d}^{\mathrm{left}} g. \tag{122}$$

It is clear that this gives a well-defined right $\star_{\mathrm{red}}$-linear $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued non-degenerate and full inner product.

Proposition 6.10 *The canonical inner product $\langle \cdot, \cdot \rangle_{\mathrm{can}}$ is completely positive.*

Proof. Let $\phi_1, \ldots, \phi_n \in \mathcal{C}^\infty_{\mathrm{cf}}(U \times G)[[\lambda]]$ and let $\Omega = \Omega_0 + \lambda\Omega_1 + \cdots$ be a positive linear functional of $M_n(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})$ such that each Ω_r is a distribution according to our convention in Remark 6.9. Then by continuity

$$\begin{aligned}
\Omega\left(\langle \phi_i, \phi_j \rangle_{\mathrm{can}}\right) &= \Omega\left(p \mapsto \int_G \left(\overline{\phi_i} \star_{\mathrm{red}} \phi_j\right)(p, g) \, \mathrm{d}^{\mathrm{left}} g\right) \\
&= \int_G \Omega\left(\overline{\phi_i}(\cdot, g) \star_{\mathrm{red}} \phi_j(\cdot, g)\right) \mathrm{d}^{\mathrm{left}} g \\
&\geq 0,
\end{aligned}$$

since the lowest non-vanishing order of the integrand is positive for every $g \in G$. □

To proceed, we need a more explicit description of $\star'$ and $\bullet'$ for our local model. Let $D \in \mathrm{DiffOp}^k_{\mathrm{ver}}(U \times G)$ be a vertical differential operator on $U \times G$. Then for a chosen basis $e_1, \ldots, e_N \in \mathfrak{g}$ we have uniquely determined functions $D^{i_1 \cdots i_N} \in \mathcal{C}^\infty(U \times G)$ such that

$$D = \sum_{|I| \leq k} D^{i_1 \cdots i_N} \left(\mathscr{L}_{(e_1)_{U \times G}}\right)^{i_1} \cdots \left(\mathscr{L}_{(e_N)_{U \times G}}\right)^{i_N}. \tag{123}$$

Since the fundamental vector fields do not commute in general, (123) can be viewed as a standard-ordered calculus with respect to the chosen basis. For abbreviation we write $e_{\vec{I}}$ for the ordered sequence of Lie derivatives in (123) and set also $D^I = D^{i_1 \cdots i_N}$ for a multiindex $I \in \mathbb{N}_0^N$. A formal series $D \in \mathrm{DiffOp}_{\mathrm{ver}}(U \times G)[[\lambda]]$ can then be written as formal series $D = \sum_I D^I e_{\vec{I}}$ with $D^I \in \mathcal{C}^\infty(U \times G)[[\lambda]]$ such that in each order of λ only finitely many differentiations occur. Up to here, this is even possible for an arbitrary principal bundle, we do not yet need the trivialization.

Only in our local model we can define now

$$D \bullet' \phi = \sum\nolimits_I D^I \star_{\mathrm{red}} e_{\vec{I}} \phi \tag{124}$$

for $D \in \mathrm{DiffOp}(U \times G)[[\lambda]]$ and $\phi \in \mathcal{C}^\infty(U \times G)[[\lambda]]$. For this action we have the following properties:

Lemma 6.11 *Let $D, \tilde{D} \in \mathrm{DiffOp}_{\mathrm{ver}}(U \times G)[[\lambda]]$, $\phi \in \mathcal{C}^\infty(U \times G)[[\lambda]]$ and $u \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$.*

i.) The definition (124) *yields a formal series of differential operators with*

$$(D \bullet' \phi) \bullet_{\mathrm{can}} u = D \bullet' (\phi \bullet_{\mathrm{can}} u). \tag{125}$$

Moreover, $\bullet'$ deforms the usual action of vertical differential operators.

ii.) There exists a unique $D \star' \tilde{D} \in \mathrm{DiffOp}_{\mathrm{ver}}(U \times G)[[\lambda]]$ such that

$$(D \star' \tilde{D}) \bullet' \phi = D \bullet' (\tilde{D} \bullet' \phi). \tag{126}$$

iii.) The product $\star'$ is the unique associative deformation of $\mathrm{DiffOp}_{\mathrm{ver}}(U \times G)[[\lambda]]$ with the unique left module structure $\bullet'$ up to equivalence such that $\mathcal{C}^\infty(U \times G)[[\lambda]]$ becomes a $(\star', \star_{\mathrm{red}})$-bimodule.

iv.) There exists a uniquely determined *-involution $D \mapsto D^*$ with respect to $\star'$ such that $\bullet'$ becomes a *-representation of the pre Hilbert module $(\mathcal{C}^\infty_{\mathrm{cf}}(U \times G)[[\lambda]], \langle \cdot, \cdot \rangle_{\mathrm{can}})$.

Proof. The property (125) is obvious by the associativity of $\star_{\mathrm{red}}$. Also $\bullet'$ deforms the usual action. For the second part we note that

$$D \bullet' (\tilde{D} \bullet' \phi) = \sum_{I,J} D^I \star_{\mathrm{red}} e_{\vec{I}} \left(\tilde{D}^J \star_{\mathrm{red}} e_{\vec{J}} \phi \right).$$

Since the fundamental vector fields $\xi_{U \times G}$ are derivations of $\star_{\mathrm{red}}$ on $U \times G$, we have a Leibniz rule allowing to redistribute the $e_{\vec{I}}$ on the two factors $\tilde{D}^J$ and $e_{\vec{J}}\phi$. In the second result we have to reorder the Lie derivatives which gives after Lie algebraic combinatorics again linear combinations of $e_{\vec{K}}\,\phi$ with *constant* coefficients. These can be viewed as acting by $\star_{\mathrm{red}}$ from the left whence in total we have by the associativity of $\star_{\mathrm{red}}$ a new $D \star' \tilde{D}$ acting via $\bullet'$ as wanted. Since by the first part the map $D \mapsto D\bullet'$ is injective, $D \star' \tilde{D}$ is uniquely determined. Note that for $i \le j$ we have

$$\mathscr{L}_{(e_i)_{U \times G}} \star' \mathscr{L}_{(e_j)_{U \times G}} = \mathscr{L}_{(e_i)_{U \times G}} \mathscr{L}_{(e_j)_{U \times G}} \quad \text{and} \quad D^I \star' e_{\vec{I}} = D^I e_{\vec{I}}.$$

Hence the fundamental vector fields and the functions $D^I \in \mathcal{C}^\infty(U \times G)[[\lambda]]$ generate (up to λ-adic completion) via $\star'$ all of $\mathrm{DiffOp}_{\mathrm{ver}}(U \times G)[[\lambda]]$. The third part is clear from general considerations on principal bundles [8]. For the last part we have to show that D is adjointable with respect to $\langle \cdot, \cdot \rangle_{\mathrm{can}}$. We do this first for the generators. If $\xi \in \mathfrak{g}$ then

$$\begin{aligned}
\pi^* \left\langle \phi, \mathscr{L}_{\xi_{U \times G}} \psi \right\rangle_{\mathrm{can}} &= \int_G \mathsf{L}_{g^{-1}} \left(\overline{\phi} \star_{\mathrm{red}} \mathscr{L}_{\xi_{U \times G}} \psi \right) \mathrm{d}^{\mathrm{left}} g \\
&= \int_G \mathsf{L}_{g^{-1}} \left(\mathscr{L}_{\xi_{U \times G}} \left(\overline{\phi} \star_{\mathrm{red}} \psi \right) - \overline{\mathscr{L}_{\xi_{U \times G}} \phi} \star_{\mathrm{red}} \psi \right) \mathrm{d}^{\mathrm{left}} g \\
&= \int_G \mathsf{L}_{g^{-1}} \left(\overline{-\Delta(\xi)\phi - \mathscr{L}_{\xi_{U \times G}} \phi} \star_{\mathrm{red}} \psi \right) \mathrm{d}^{\mathrm{left}} g \\
&= \pi^* \left\langle \left(-\Delta(\xi) - \mathscr{L}_{\xi_{U \times G}} \right) \bullet' \phi, \psi \right\rangle_{\mathrm{can}}.
\end{aligned}$$

Thus we have an adjoint in $\mathrm{DiffOp}_{\mathrm{ver}}(U \times G)[[\lambda]]$. Analogously, we have for a function $D^I \in \mathcal{C}^\infty(U \times G)[[\lambda]]$

$$\begin{aligned}
\pi^* \left\langle \phi, D^I \bullet' \psi \right\rangle_{\mathrm{can}} &= \int_G \mathsf{L}_{g^{-1}} \left(\overline{\phi} \star_{\mathrm{red}} (D^I \star_{\mathrm{red}} \psi) \right) \mathrm{d}^{\mathrm{left}} g \\
&= \int_G \mathsf{L}_{g^{-1}} \left(\overline{(\overline{D^I} \star_{\mathrm{red}} \phi)} \star_{\mathrm{red}} \psi \right) \mathrm{d}^{\mathrm{left}} g \\
&= \pi^* \left\langle \overline{D^I} \bullet' \phi, \psi \right\rangle_{\mathrm{can}},
\end{aligned}$$

since $\star_{\mathrm{red}}$ is Hermitian. Successively using these two statements and the fact that these generate all vertical differential operators, proves that all $D \in \mathrm{DiffOp}_{\mathrm{ver}}(U \times G)[[\lambda]]$ have an adjoint in the vertical differential operators. Thus the last part follows. □

On C we consider now the following type of inner product: let $\langle \cdot, \cdot \rangle'$ be a $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued inner product on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ such that there exists a formal series $B = B_0 + \lambda B_1 + \cdots$ of bidifferential operators on C with

$$\pi^* \langle \phi, \psi \rangle' = \int_G \mathsf{L}_{g^{-1}} \left(B(\overline{\phi}, \psi) \right) \mathrm{d}^{\mathrm{left}} g, \tag{127}$$

and $B_0(\overline{\phi}, \psi) = \overline{\phi}\psi$. In this case we call $\langle \cdot, \cdot \rangle'$ a *bidifferential deformation* of the canonical classical inner product (114). Note that B is not uniquely determined by (127) since we can still perform integrations by parts. For our local situation we have now the following result:

Lemma 6.12 *Let $\langle\cdot,\cdot\rangle'$ be a bidifferential deformation of the canonical classical inner product on $\mathcal{C}^\infty_{\mathrm{cf}}(U\times G)[[\lambda]]$.*

i.) There exist $B^{IJ}\in\mathcal{C}^\infty(U\times G)[[\lambda]]$ such that

$$\pi^*\langle\phi,\psi\rangle' = \sum_{I,J}\int_G \mathsf{L}_{g^{-1}}\left(e_{\vec{I}}(\overline{\phi})\star_{\mathrm{red}} B^{IJ}\star_{\mathrm{red}} e_{\vec{J}}(\psi)\right)\mathrm{d}^{\mathrm{left}} g, \tag{128}$$

where the sum is infinite but in each order of λ we have only finitely many differentiations.

ii.) There exists a vertical differential operator $H\in\mathrm{DiffOp}_{\mathrm{ver}}(U\times G)[[\lambda]]$ such that $H=H^*=\mathsf{id}+\cdots$ and

$$\langle\phi,\psi\rangle' = \langle\phi, H\bullet'\psi\rangle_{\mathrm{can}}. \tag{129}$$

iii.) $\langle\cdot,\cdot\rangle'$ is isometric to $\langle\cdot,\cdot\rangle_{\mathrm{can}}$.

Proof. For the first part consider $\phi=u\otimes\chi$ and $\psi=v\otimes\tilde{\chi}$ with $u,v\in\mathcal{C}^\infty(U)$ and $\chi,\tilde{\chi}\in\mathcal{C}^\infty_0(G)$. Using the right $\star_{\mathrm{red}}$-linearity of $\langle\cdot,\cdot\rangle'$ we get

$$\int_G \mathsf{L}_{g^{-1}}\left(B(\overline{u\otimes\chi}, v\otimes\tilde{\chi})\right)\mathrm{d}^{\mathrm{left}} g = \pi^*(\overline{u})\star_{\mathrm{red}}\int_G \mathsf{L}_{g^{-1}}\left(B(1\otimes\overline{\chi},1\otimes\tilde{\chi})\right)\mathrm{d}^{\mathrm{left}} g\star_{\mathrm{red}}\pi^*(v).$$

Now in $B(1\otimes\overline{\chi},1\otimes\tilde{\chi})$ only vertical differentiations can contribute. Hence we have

$$B(1\otimes\overline{\chi},1\otimes\tilde{\chi}) = \sum_{I,J} e_{\vec{I}}(1\otimes\overline{\chi})B^{IJ}e_{\vec{J}}(1\otimes\tilde{\chi})$$

with formal series $B^{IJ}\in\mathcal{C}^\infty(U\times G)[[\lambda]]$ such that in each order of λ only finitely many differentiations occur. Since $\pi^*(\overline{u})$ and π^*v do not depend on the group variables and since the fundamental vector fields are derivations of $\star_{\mathrm{red}}$ we arrive at the formula

$$\pi^*\langle u\otimes\chi, v\otimes\tilde{\chi}\rangle' = \sum_{I,J}\int_G \mathsf{L}_{g^{-1}}\left(\overline{e_{\vec{I}}(u\otimes\chi)}\star_{\mathrm{red}} B^{IJ}\star_{\mathrm{red}} e_{\vec{J}}(v\otimes\tilde{\chi})\right)\mathrm{d}^{\mathrm{left}} g.$$

Now in each order of λ we have an integration and bidifferential operators. By the usual continuity and density argument, they are already determined on their values on factorizing functions $u\otimes\chi$ and $v\otimes\tilde{\chi}$, respectively. Thus (128) holds in general showing the first part. Since the $e_{\vec{I}}$ are real differential operators, we can rewrite this as

$$\langle\phi,\psi\rangle' = \sum_{I,J}\langle e_{\vec{I}}\bullet'\phi, (B^{IJ}e_{\vec{J}})\bullet'\psi\rangle_{\mathrm{can}} = \left\langle\phi, \left(\sum_{I,J} e^*_{\vec{I}}\star'(B^{IJ}e_{\vec{J}})\right)\bullet'\psi\right\rangle_{\mathrm{can}}.$$

This yields the vertical differential operator $H\in\mathrm{DiffOp}_{\mathrm{ver}}(U\times G)[[\lambda]]$. From $\langle\phi,H\bullet'\psi\rangle_{\mathrm{can}} = \langle\phi,\psi\rangle' = \overline{\langle\psi,\phi\rangle'} = \overline{\langle\psi,H\bullet'\phi\rangle_{\mathrm{can}}} = \langle H\bullet'\phi,\psi\rangle_{\mathrm{can}}$ we see $H=H^*$. Finally, $H=\mathsf{id}+\cdots$ is clear giving the second part. The third part follows as we have a Hermitian $\star'$-square root $\sqrt[\star']{H}\star'\sqrt[\star']{H}=H$ which implements the unitary map between the two inner products by left $\bullet'$-multiplication. □

Corollary 6.13 *Every bidifferential deformation of $\langle\cdot,\cdot\rangle^{\mathrm{cl}}_{\mathrm{can}}$ on $\mathcal{C}^\infty_{\mathrm{cf}}(U\times G)[[\lambda]]$ is completely positive.*

After these local constructions we shall now pass to the global situation. The next proposition gives the existence of $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued inner products which deform the canonical classical one in a bidifferential way. Of course, our inner product $\langle\cdot,\cdot\rangle_{\mathrm{red}}$ is of this form. However, we give an independent proof not relying on phase space reduction thereby including non-connected Lie groups G as well.

Proposition 6.14 *Let $C \circlearrowleft G \longrightarrow M_{\mathrm{red}}$ be an arbitrary principal bundle. Then there exists a bidifferential deformation $\langle\cdot,\cdot\rangle$ of the canonical classical inner product on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ with the additional feature*

$$\left\langle \mathsf{L}_{g^{-1}}\phi, \mathsf{L}_{g^{-1}}\psi \right\rangle = \Delta(g)\,\langle\phi,\psi\rangle \tag{130}$$

for all $\phi,\psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ and $g \in G$.

Proof. Let $\{U_\alpha, \Phi_\alpha\}$ be again a locally finite atlas of trivializations and let $\{\chi_\alpha\}$ be a subordinate quadratic partition of unity on M_{red}, i.e. $\operatorname{supp}\chi_\alpha \subseteq U_\alpha$ and $\sum_\alpha \overline{\chi_\alpha}\chi_\alpha = 1$. The global right module structure $\bullet_{\mathrm{red}}$ of $\mathcal{C}^\infty(C)[[\lambda]]$ restricts to $\pi^{-1}(U_\alpha)$ and, via Φ_α we obtain a right module structure $\bullet_\alpha$ on each $\mathcal{C}^\infty(U_\alpha \times G)[[\lambda]]$, i.e. we have

$$\phi \bullet_\alpha u = (\Phi_\alpha^*)^{-1}\left(\Phi_\alpha^*\phi \bullet_{\mathrm{red}} u\right)$$

for $\phi \in \mathcal{C}^\infty(U_\alpha \times G)[[\lambda]]$ and $u \in \mathcal{C}^\infty(U_\alpha)[[\lambda]]$. By the uniqueness of the right module structure we find a G-equivariant formal series of differential operators $T_\alpha = \mathsf{id} + \sum_{r=1}^\infty \lambda^r T_\alpha^{(r)}$ on $U_\alpha \times G$ such that

$$T_\alpha(\phi \bullet_\alpha u) = T_\alpha(\phi) \star_{\mathrm{red}} \pi^* u.$$

Here we use again $\star_{\mathrm{red}}$ also for $\mathcal{C}^\infty(U_\alpha \times G)[[\lambda]]$ making π^* a star product homomorphism as in (121). This way, we define an inner product on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ by

$$\langle\phi,\psi\rangle = \sum_\alpha \left\langle \pi^*\chi_\alpha \star_{\mathrm{red}} T_\alpha((\Phi_\alpha^*)^{-1}\phi), \pi^*\chi_\alpha \star_{\mathrm{red}} T_\alpha((\Phi_\alpha^*)^{-1}\psi)\right\rangle_{\mathrm{can}}. \tag{$*$}$$

Indeed, $\langle\phi,\psi\rangle \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ is well-defined since $T_\alpha((\Phi_\alpha^*)^{-1}\phi)$, $T_\alpha((\Phi_\alpha^*)^{-1}\psi) \in \mathcal{C}^\infty_{\mathrm{cf}}(U_\alpha \times G)[[\lambda]]$ become globally defined functions on $M_{\mathrm{red}} \times G$ after multiplying with $\pi^*\chi_\alpha$ thanks to $\operatorname{supp}\chi_\alpha \subseteq U_\alpha$ and the fact that $\star_{\mathrm{red}}$ is bidifferential. Then each term in the above sum has support in the appropriate U_α. By the local finiteness of the cover, $(*)$ is well-defined and smooth in each order of λ. The $\mathbb{C}[[\lambda]]$-sesquilinearity and the symmetry under complex conjugation is clear as $\langle\cdot,\cdot\rangle_{\mathrm{can}}$ has these features and all involved maps are $\mathbb{C}[[\lambda]]$-linear. Now let $u \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ then we have

$$\begin{aligned}
\pi^*\chi_\alpha \star_{\mathrm{red}} T_\alpha\left((\Phi_\alpha^*)^{-1}(\psi \bullet_{\mathrm{red}} u)\right) &= \pi^*\chi_\alpha \star_{\mathrm{red}} T_\alpha\left((\Phi_\alpha^*)^{-1}\psi \bullet_\alpha u)\right) \\
&= \pi^*\chi_\alpha \star_{\mathrm{red}} \left(T_\alpha((\Phi_\alpha^*)^{-1}\psi) \star_{\mathrm{red}} \pi^* u\right) \\
&= \left(\pi^*\chi_\alpha \star_{\mathrm{red}} T_\alpha((\Phi_\alpha^*)^{-1}\psi)\right) \star_{\mathrm{red}} \pi^* u.
\end{aligned}$$

Since $\langle\cdot,\cdot\rangle_{\mathrm{can}}$ is right $\star_{\mathrm{red}}$-linear in the second argument we deduce $\langle\phi, \psi \bullet_{\mathrm{red}} u\rangle = \langle\phi,\psi\rangle \star_{\mathrm{red}} u$. Thus $\langle\cdot,\cdot\rangle$ is indeed a valid inner product. We compute its classical limit. Since T_α is the identity in the zeroth order of λ we get

$$\begin{aligned}
\langle\phi,\psi\rangle &= \sum_\alpha \left\langle \pi^*\chi_\alpha(\Phi_\alpha^*)^{-1}\phi, \pi^*\chi_\alpha(\Phi_\alpha^*)^{-1}\psi\right\rangle_{\mathrm{can}} + \cdots \\
&= \sum_\alpha \int_G \mathsf{L}_{g^{-1}}\left((\Phi_\alpha^*)^{-1}\left(\overline{\tilde{\chi}_\alpha\phi}\tilde{\chi}_\alpha\psi\right)\right) \mathrm{d}^{\mathrm{left}} g + \cdots \\
&= \sum_\alpha (\Phi_\alpha^*)^{-1}\left(\int_G \mathsf{L}_{g^{-1}}\left(\overline{\tilde{\chi}_\alpha\phi}\tilde{\chi}_\alpha\psi\right) \mathrm{d}^{\mathrm{left}} g\right) + \cdots \\
&= \int_G \mathsf{L}_{g^{-1}}\left(\overline{\phi}\psi\right) \mathrm{d}^{\mathrm{left}} g + \cdots,
\end{aligned}$$

where we set $\tilde{\chi}_\alpha = \Phi_\alpha^* \pi^* \chi_\alpha$ which yields a partition of unity on C subordinate to the cover $\{\pi^{-1}(U_\alpha)\}$ with $\sum_\alpha \overline{\tilde{\chi}_\alpha} \tilde{\chi}_\alpha = 1$. The last equation follows since the integral is already an invariant function which can directly be identified with a function on U_α, not needing the trivialization anymore. Thus we have a deformation as wanted. Finally, let $g \in G$. Since all the maps T_α and Φ_α^* are equivariant and $\mathsf{L}_{g^{-1}} \pi^* \chi_\alpha = \pi^* \chi_\alpha$, we get

$$\pi^* \chi_\alpha \star_{\mathrm{red}} T_\alpha((\Phi_\alpha^*)^{-1}(\mathsf{L}_{g^{-1}} \psi)) = \pi^* \chi_\alpha \star_{\mathrm{red}} \left(\mathsf{L}_{g^{-1}} T_\alpha((\Phi_\alpha^*)^{-1} \phi) \right) = \mathsf{L}_{g^{-1}} \left(\pi^* \chi_\alpha \star_{\mathrm{red}} T_\alpha((\Phi_\alpha^*)^{-1} \phi) \right).$$

Now the locally defined $\langle \cdot, \cdot \rangle_{\mathrm{can}}$ has the property (130) and hence $\langle \cdot, \cdot \rangle$ inherits this since every term in $(*)$ satisfies (130). □

In the last step, we show three things: for a given bidifferential deformation $\langle \cdot, \cdot \rangle$ of the classical canonical inner product the vertical differential operators act in an adjointable way, all such deformations are completely positive and isometric:

Theorem 6.15 *Let $C \circlearrowright G \longrightarrow M_{\mathrm{red}}$ be an arbitrary principal bundle. Moreover, let*

$$\langle \cdot, \cdot \rangle : \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]] \times \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]] \longrightarrow \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]] \tag{131}$$

be a bidifferential deformation of the canonical classical inner product with respect to a given right module structure $\bullet_{\mathrm{red}}$. Moreover, let $\star'$ be a corresponding choice of a deformation of the vertical differential operators $\mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$ with left module structure $\bullet'$.

i.) There exists a unique *-involution for $(\mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]], \star')$ deforming the classical one such that $\bullet'$ becomes a *-representation with respect to $\langle \cdot, \cdot \rangle$.

ii.) The inner product $\langle \cdot, \cdot \rangle$ is completely positive.

iii.) Any two deformations are isometrically isomorphic via the left $\bullet'$-multiplication of some $V = \mathsf{id} + \sum_{r=1}^\infty \lambda^r V_r \in \mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$.

Proof. As before, we choose a locally finite atlas $\{(U_\alpha, \Phi_\alpha)\}$ of trivializations and a subordinate partition of unity $\{\chi_\alpha\}$. For $D \in \mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$ and $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ we have

$$\pi^* \langle \phi, D \bullet' \psi \rangle = \int_G \mathsf{L}^*_{g^{-1}} \left(B(\overline{\phi}, D \bullet' \psi) \right) \mathrm{d}^{\mathrm{left}} g = \sum_\alpha \int_G \mathsf{L}^*_{g^{-1}} \left(B\left(\overline{\phi}, (\pi^* \chi_\alpha D) \bullet' \psi\right) \right) \mathrm{d}^{\mathrm{left}} g.$$

Since B is bidifferential and $\bullet'$ is also local, we have

$$\operatorname{supp} B(\overline{\phi}, (\pi^* \chi_\alpha D) \bullet' \psi) \subseteq \operatorname{supp} \pi^* \chi_\alpha \subseteq \pi^{-1}(U_\alpha).$$

It follows that $\langle \phi, (\pi^* \chi_\alpha D) \bullet' \psi \rangle$ is given by the restriction of $\langle \cdot, \cdot \rangle$ to $\mathcal{C}^\infty_{\mathrm{cf}}(\pi^{-1}(U_\alpha))[[\lambda]]$ evaluated on the restrictions of ϕ and $(\pi^* \chi_\alpha D) \bullet' \psi$, respectively. Here we can apply Lemma 6.12, *iii.)*, and find an isometry $V_\alpha = \mathsf{id} + \sum_{r=1}^\infty \lambda^r V_\alpha^{(r)} \in \mathrm{DiffOp}_{\mathrm{ver}}(\pi^{-1}(U_\alpha))[[\lambda]]$ such that

$$\left\langle \phi, (\pi^* \chi_\alpha D) \bullet' \psi \right\rangle = \left\langle V_\alpha \bullet' \phi, V_\alpha \bullet' \left((\pi^* \chi_\alpha D) \bullet' \psi \right) \right\rangle_{\mathrm{can}}.$$

With respect to the locally defined canonical inner product, the action of the vertical differential operators is adjointable according to Lemma 6.11: there we have shown this for a particular choice of $\bullet'$ but all these choices are equivalent which allows to transport the *-involution from the particular choice to any other $\star'$ and $\bullet'$. This way, we get a locally defined *-involution $^{*_\alpha}$ for $\star'$ compatible with $\bullet'$ and $\langle \cdot, \cdot \rangle_{\mathrm{can}}$. Using the invertibility of $H_\alpha = V_\alpha^{*_\alpha} \star' V_\alpha = \mathsf{id} + \cdots$ as before, we get

$$\begin{aligned}
\left\langle \phi, (\pi^* \chi_\alpha D) \bullet' \psi \right\rangle &= \left\langle V_\alpha \bullet' \phi, V_\alpha \bullet' (\pi^* \chi_\alpha D) \bullet' \psi \right\rangle_{\mathrm{can}} \\
&= \left\langle H_\alpha \bullet' \phi, (\pi^* \chi_\alpha D) \bullet' \psi \right\rangle_{\mathrm{can}} \\
&= \left\langle \left((\pi^* \chi_\alpha D)^{*_\alpha} \star' H_\alpha \right) \bullet' \phi, \psi \right\rangle_{\mathrm{can}}
\end{aligned}$$

$$= \left\langle V_\alpha \bullet' \left(H_\alpha^{-1} \star' (\pi^*\chi_\alpha D)^{*_\alpha} \star' H_\alpha \right) \bullet' \phi, V_\alpha \bullet' \psi \right\rangle_{\mathrm{can}}$$

$$= \left\langle \left(H_\alpha^{-1} \star' (\pi^*\chi_\alpha D)^{*_\alpha} \star' H_\alpha \right) \bullet' \phi, \psi \right\rangle .$$

Since all the operations $\star'$ and $\bullet'$ preserve the supports we can finally take the sum over all α and get

$$\left\langle \phi, D \bullet' \psi \right\rangle = \sum_\alpha \left\langle \phi, (\pi^*\chi_\alpha D) \bullet' \psi \right\rangle = \left\langle \left(\sum_\alpha H_\alpha^{-1} \star' (\pi^*\chi_\alpha D) \star' H_\alpha \right) \bullet' \phi, \psi \right\rangle = \langle D^* \bullet \phi, \psi \rangle ,$$

with $D^* \in \mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$ according to the term before. This shows that we indeed obtain an adjoint for the left action of D. Since $\langle \cdot, \cdot \rangle$ is in zeroth order just the canonical classical inner product, the classical limit of the *-involution is the classical *-involution. Since $\langle \cdot, \cdot \rangle$ is non-degenerate and $D \mapsto (\phi \mapsto D \bullet' \phi)$ is injective, the *-involution is necessarily unique, proving the first part. The second part is now very easy: using a quadratic partition of unity $\sum_\alpha \overline{\chi_\alpha} \chi_\alpha = 1$ subordinate to the above atlas, we obtain vertical differential operators $\boldsymbol{\chi}_\alpha \in \mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$ with

$$\sum_\alpha \boldsymbol{\chi}_\alpha^* \star' \boldsymbol{\chi}_\alpha = \mathsf{id},$$

with $\operatorname{supp} \boldsymbol{\chi}_\alpha \subseteq \pi^{-1}(U_\alpha)$ and $\boldsymbol{\chi}_\alpha = \pi^*\chi_\alpha + \cdots$. Indeed, the vertical differential operator $X = \sum_\alpha (\pi^*\chi_\alpha)^* \star' \pi^*\chi_\alpha = \mathsf{id} + \cdots$ is Hermitian and starts with the identity, since the classical limit of the *-involution is just the complex conjugation on $\pi^*\chi_\alpha$, viewed as vertical differential operator. Thus $\sqrt[\star']{X} = \mathsf{id} + \cdots$ is well-defined and invertible. Then $\boldsymbol{\chi}_\alpha = \pi^*\chi_\alpha \star' \frac{1}{\sqrt[\star']{X}}$ will do the job. Using this, we get for $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$

$$\langle \phi, \psi \rangle = \left\langle \phi, \sum_\alpha \left(\boldsymbol{\chi}_\alpha^* \star' \boldsymbol{\chi}_\alpha \right) \bullet' \psi \right\rangle = \sum_\alpha \langle \boldsymbol{\chi}_\alpha \bullet \phi, \boldsymbol{\chi}_\alpha \bullet \psi \rangle \qquad (*)$$

with $\boldsymbol{\chi}_\alpha \bullet \phi, \boldsymbol{\chi}_\alpha \bullet \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(\pi^{-1}(U_\alpha))[[\lambda]]$. Here we can apply Corollary 6.13 to get the complete positivity locally, and, since we have a (locally finite) convex sum in $(*)$, also globally. Thus the second part follows. For the third, let $\langle \cdot, \cdot \rangle'$ be another inner product. Then they are isometric on $\pi^{-1}(U_\alpha)$ via some isometry $V_\alpha = \mathsf{id} + \cdots \in \mathrm{DiffOp}_{\mathrm{ver}}(\pi^{-1}(U_\alpha))[[\lambda]]$, i.e. $\langle \phi, \psi \rangle = \langle V_\alpha \bullet' \phi, V_\alpha \bullet' \psi \rangle'$ for all $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(\pi^{-1}(U_\alpha))[[\lambda]]$, according to Lemma 6.12. We apply this to $(*)$ and get for arbitrary $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$

$$\begin{aligned}
\langle \phi, \psi \rangle &= \sum_\alpha \left\langle \boldsymbol{\chi}_\alpha \bullet' \phi, \boldsymbol{\chi}_\alpha \bullet' \psi \right\rangle \\
&= \sum_\alpha \left\langle V_\alpha \bullet' (\boldsymbol{\chi}_\alpha \bullet' \phi), V_\alpha \bullet' (\boldsymbol{\chi}_\alpha \bullet' \psi) \right\rangle' \\
&= \left\langle \phi, \left(\sum_\alpha \boldsymbol{\chi}_\alpha^{*'} \star' V_\alpha^{*'} \star' V_\alpha \star' \boldsymbol{\chi}_\alpha \right) \bullet' \psi \right\rangle' \\
&= \left\langle \phi, H \bullet' \psi \right\rangle',
\end{aligned}$$

with some $H \in \mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$ given explicitly by the locally finite sum

$$H = \sum_\alpha \boldsymbol{\chi}_\alpha^{*'} \star' V_\alpha^{*'} \star' V_\alpha \star' \boldsymbol{\chi}_\alpha,$$

where $^{*'}$ denotes the *-involution induced by $\langle \cdot, \cdot \rangle'$ according to the first part. From the construction it is clear that $H = \mathsf{id} + \cdots$. Thus we have $H = V^{*'} \star' V$ with some $V = \mathsf{id} + \cdots \in \mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$ which is the isometry we are looking for. □

Remark 6.16 This result, together with the existence according to Proposition 6.14, can be seen as an extension of the (rigidity) results from [8] on the existence and uniqueness of the right module structure $\bullet_{\mathrm{red}}$: also the canonical classical inner product allows for an essentially unique deformation preserving complete positivity and the adjointability of the vertical differential operators. Note also, that the above construction is independent of the phase space reduction approach, which also gives existence of an inner product but no proof for uniqueness. Moreover, in the phase space reduction approach we are restricted to principal bundles arising from *connected* groups.

6.4 A strong Morita equivalence bimodule

We can now formulate the main result of this section, the quantized version of Theorem 6.5:

Theorem 6.17 *Let $C \circlearrowleft G \longrightarrow M_{\mathrm{red}}$ be an arbitrary principal bundle and $\langle \cdot, \cdot \rangle$ a bidifferential deformation of $\langle \cdot, \cdot \rangle^{\mathrm{cl}}_{\mathrm{red}}$.*

i.) The inner product $\langle \cdot, \cdot \rangle$ is full, completely positive, and there is a $\boldsymbol{e} \in \mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ with

$$\langle \boldsymbol{e}, \boldsymbol{e} \rangle = 1. \tag{132}$$

ii.) The canonical inner product $\Theta_{\cdot,\cdot}$ with values in the finite rank operators $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]])$ is completely positive as well.

iii.) $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ is a strong Morita equivalence bimodule for $\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]])$ and $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ deforming the classical strong Morita equivalence bimodule $\mathcal{C}^\infty_{\mathrm{cf}}(C)$ from Theorem 6.5.

Proof. The first part is now clear from Theorem 6.15, *ii.)*, and an argument analogous to the one in Proposition 6.2. Then the second part follows as in Proposition 6.2, too, which gives the last part immediately. □

The deformed vertical differential operators $\mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$ are not (strongly) Morita equivalent to $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$, neither is $(\mathcal{C}^\infty(M)[[\lambda]], \star)$, see Remark 6.4, in the case of phase space reduction. On the other hand, these algebras are not very far away from being strongly Morita equivalent to $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$, since we have a strong Morita equivalence bimodule and a *-homomorphism into the adjointable operators. The only flaw is that this *-homomorphism does not map into the finite rank operators. Note that in the case of $\mathrm{DiffOp}_{\mathrm{ver}}(C)[[\lambda]]$ it is even injective, while for $\mathcal{C}^\infty(M)[[\lambda]]$ we clearly loose the functions with vanishing infinite jet at C.

Again, we have a very rigid situation for the deformation of the inner products and the bimodule structure as already for the strong Morita equivalence bimodules in deformation quantization of *unital* algebras, see [10]. In our case, the crucial new feature is that one of the algebras is non-unital.

Remark 6.18 (Rieffel induction) Having the strong Morita equivalence bimodule we obtain by *Rieffel induction* an equivalence of categories

$$\begin{gathered}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]], \bullet_{\mathrm{red}}) \mathbin{\widehat{\otimes}}_{(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})} \cdot : \\ {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathfrak{F}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]], \bullet_{\mathrm{red}}))\end{gathered} \tag{133}$$

for every coefficient *-algebra $\mathcal{D}$, see (111). Moreover, since also the *-algebra $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ acts on $\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]]$ via $\bullet$ in an adjointable way thanks to Proposition 5.11 we obtain also a Rieffel induction functor

$$\begin{gathered}(\mathcal{C}^\infty_{\mathrm{cf}}(C)[[\lambda]], \bullet_{\mathrm{red}}) \mathbin{\widehat{\otimes}}_{(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}})} \cdot : \\ {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]], \star_{\mathrm{red}}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{C}^\infty(M)[[\lambda]], \star).\end{gathered} \tag{134}$$

However, in general this will not be an equivalence of categories anymore. The reason is clear from geometric considerations: Indeed, the image of a *-representation of $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ under (134) is somehow located on C, in the sense that if f vanishes on C up to infinite order, then the action of f in an induced representation is necessarily trivial. This is clear from the bidifferentiality of the left module structure $\bullet$. On the other hand, $\mathcal{C}^\infty(M)[[\lambda]]$ does have non-trivial *-representations located away from C: we can take any δ-functional at $p \in M \setminus C$ and deform it into a positive functional ω_p with support still be given by p. Then the GNS representation π_{ω_p} of ω_p is not the trivial representation. In fact, since the deformation $\omega_p = \delta_p \circ S$ is obtained by means of a formal series $S_p = \mathsf{id} + \sum_{r=1}^\infty \lambda^r S_r$ with differential operators S_r vanishing on constants, a function $f \in \mathcal{C}^\infty(M)[[\lambda]]$ which is 1 in an open neighbourhood of p acts as identity operator on the GNS pre Hilbert space of ω_p. However, considering a function f which vanishes up to infinite order on C, we conclude that the GNS representation can not be in the image of (134) up to unitary equivalence. This shows that (134) will not be an equivalence of categories, see also Remark 6.4.

7 An example

In this concluding section we consider the geometrically trivial situation $M = M_{\mathrm{red}} \times T^*G$ where on M_{red} a Poisson bracket and a corresponding star product $\star_{\mathrm{red}}$ is given while on T^*G we use the canonical symplectic Poisson structure and the canonical star product $\star_{\mathrm{G}}$ from [18]. Then M carries the star product $\star = \star_{\mathrm{red}} \otimes \star_{\mathrm{G}}$. Classically, the phase space reduction for the constraint hypersurface $C = M_{\mathrm{red}} \times G$ will just omit the factor T^*G and reproduces M_{red}.

7.1 The reduction from $M_{\mathrm{red}} \times T^*G$ to M_{red}

Let $\iota : G \longrightarrow T^*G$ denote the zero section of the cotangent bundle and $\mathrm{pr} : T^*G \longrightarrow G$ the bundle projection. We use the same symbols for the corresponding maps $\iota : C = M_{\mathrm{red}} \times G \longrightarrow M$ and $\mathrm{pr} : M \longrightarrow M_{\mathrm{red}} \times G$. Then C is clearly coisotropic in M with corresponding orbit space M_{red}, reproducing the given Poisson structure. In principle, one does not need the group structure of G for this coisotropic reduction; it would work literally the same for any cotangent bundle. However, in view of our previous framework, we shall outline the underlying symmetry structure.

In order to be conform with the local models described in Section 6 and the appendix we choose the *right* multiplications $\mathsf{r} : G \times G \longrightarrow G$ as group action of G on itself. The canonical lift to a *left* action on T^*G is then denoted by $\mathsf{L} : G \times T^*G \longrightarrow T^*G$, i.e. $\mathsf{L}_g = T^*\mathsf{r}_g$. This extends to M in the usual way yielding a Poisson action of G on M. The fundamental vector fields of r are the left invariant vector fields $X_\xi(g) = T_e\mathsf{l}_g(\xi)$ for $\xi \in \mathfrak{g}$ and $g \in G$. More precisely, $\xi_G = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathsf{r}^{-1}_{\exp(t\xi)} = -X_\xi$ since we defined the fundamental vector field with respect to the left action, see (7).

An effective description of the corresponding fundamental vector fields on T^*G and M are obtained as follows. To every vector field $X \in \Gamma^\infty(TG)$ we assign a fiberwisely linear function $\mathcal{J}(X) \in \mathrm{Pol}^1(T^*G)$ on T^*G by $\mathcal{J}(X)(\alpha_g) = \alpha_g(X(g))$ where $\alpha_g \in T^*_gG$. Then the canonical Poisson bracket on T^*G of such linear functions in the “momenta” is $\{\mathcal{J}(X), \mathcal{J}(Y)\} = -\mathcal{J}([X,Y])$, see also [27, Sect. 3.3.1]. Then the fundamental vector fields of the left action $\mathsf{L} = T^*\mathsf{r}$ are given by the Hamiltonian vector fields $\xi_{T^*G} = -X_{\mathcal{J}(X_\xi)}$. Thus the momentum map is given by $J(\xi) = -\mathcal{J}(X_\xi)$, which induces also the trivialization of the global tubular neighbourhood $M_{\mathrm{red}} \times T^*G$ of $M_{\mathrm{red}} \times G$. The prolongation with respect to this tubular neighbourhood according to (14) is then just the pull-back $\mathrm{prol} = \mathrm{pr}^*$.

To describe the classical Koszul operator and the homotopy more explicitly, we make use of a vector space basis $e_1, \ldots, e_N \in \mathfrak{g}$ as before. We have the corresponding left invariant vector fields $X_a = X_{e_a}$ yielding the linear functions $J_a = -P_a = -\mathcal{J}(X_a) \in \mathrm{Pol}^1(T^*G)$ in the momenta. For a one-form $\theta \in \Gamma^\infty(T^*G)$ we have the vertical lift $\theta^{\mathrm{ver}} \in \Gamma^\infty(T(T^*G))$ to a vertical vector field on T^*G. In particular, the left invariant one-forms $\theta^a \in \Gamma^\infty(T^*G)$ with value e^a at $e \in G$ lift to vertical vector fields denoted by $\frac{\partial}{\partial J_a} = -\frac{\partial}{\partial P_a} = -(\theta^a)^{\mathrm{ver}}$. Indeed, we have $\frac{\partial}{\partial J_a} J_b = \delta_b^a$, explaining our notation. The funny minus sign is due to our previous convention on fundamental vector fields.

The classical Koszul operator will then be given by $\partial x = \mathrm{i}(e^a) x J_a$ as before and the homotopy h_0 is explicitly and globally given by

$$(h_0 f)(p, \alpha_g) = e^a \int_0^1 \frac{\partial f}{\partial J_a}(p, t\alpha_g)\, \mathrm{d}\, t, \tag{135}$$

where $p \in M_{\mathrm{red}}$ and $\alpha_g \in T_g^* G$ as before.

7.2 The canonical star product $\star_{\mathrm{G}}$ and its Schrödinger representation

On T^*G there is a canonical star product $\star_{\mathrm{G}}$ which can be obtained as follows, see [18] as well as [7].

For left invariant vector fields $X_{\xi_1}, \ldots X_{\xi_k} \in \Gamma^\infty(T^*G)$ and a function $\phi \in \mathcal{C}^\infty(G)$ we define the standard ordered quantization map ϱ_{Std} by

$$\varrho_{\mathrm{Std}}\left(\mathrm{pr}^* \phi \mathcal{J}(X_{\xi_1}) \cdots \mathcal{J}(X_{\xi_k})\right) \psi = \frac{1}{k!} \left(\frac{\lambda}{\mathrm{i}}\right)^r \phi \sum_{\sigma \in S_k} \mathscr{L}_{X_{\xi_{\sigma(1)}}} \cdots \mathscr{L}_{X_{\xi_{\sigma(k)}}} \psi, \tag{136}$$

where $\psi \in \mathcal{C}^\infty(G)$, and extend this to a $\mathbb{C}[[\lambda]]$-linear map

$$\varrho_{\mathrm{Std}} : \mathrm{Pol}^\bullet(T^*G)[[\lambda]] \longrightarrow \mathrm{DiffOp}(G)[[\lambda]]. \tag{137}$$

Clearly, at this stage for polynomial functions we have convergence in λ for trivial reasons. Setting $\lambda = \hbar > 0$ yields a symbol calculus for differential operators on G and symbols on T^*G which are polynomial in the fibers.

Alternatively, the above quantization can also be written as

$$\varrho_{\mathrm{Std}}(f)\psi = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\lambda}{\mathrm{i}}\right)^r \sum_{a_1, \ldots, a_r} \iota^* \left(\frac{\partial^r f}{\partial P_{a_1} \cdots \partial P_{a_r}}\right) \mathscr{L}_{X_{a_1}} \cdots \mathscr{L}_{X_{a_r}} \psi. \tag{138}$$

Now it is clear that ϱ_{Std} extends to a $\mathbb{C}[[\lambda]]$-linear map $\varrho_{\mathrm{Std}} : \mathcal{C}^\infty(T^*G)[[\lambda]] \longrightarrow \mathrm{DiffOp}(G)[[\lambda]]$ by the very same formula as above.

Remark 7.1 The above symbol calculus can also be obtained from a covariant derivative, namely the "half-commutator connection" on G which is defined by $\nabla_{X_\xi} X_\eta = \frac{1}{2}[X_\xi, X_\eta]$ on left invariant vector fields. This point of view was taken in [7].

Clearly, for $f, g \in \mathrm{Pol}^\bullet(T^*G)[[\lambda]]$ there is a unique $f \star_{\mathrm{Std}} g \in \mathrm{Pol}^\bullet(T^*G)[[\lambda]]$ with

$$\varrho_{\mathrm{Std}}(f \star_{\mathrm{Std}} g) = \varrho_{\mathrm{Std}}(f)\, \varrho_{\mathrm{Std}}(g). \tag{139}$$

Moreover, this extends to a bidifferential star product for arbitrary $f, g \in \mathcal{C}^\infty(T^*G)[[\lambda]]$ preserving (139). This star product is *standard-ordered* in the sense that $\mathrm{pr}^* \phi \star_{\mathrm{Std}} f = \mathrm{pr}^* \phi f$ for arbitrary $\phi \in \mathcal{C}^\infty(G)[[\lambda]]$.

The only flaw of $\star_{\mathrm{Std}}$ is that it is not Hermitian. This can be understood and cured as follows. First we introduce the differential operator

$$\Delta_0 = \mathscr{L}_{X_{P_a}} \mathscr{L}_{\frac{\partial}{\partial P_a}} \tag{140}$$

acting on functions on T^*G, where as before X_{P_a} is the Hamiltonian vector field of the global momentum function P_a. Clearly, this operator is independent of the chosen basis. Moreover, we need the vertical lift of the modular one-form Δ which yields the vector field $\Delta^{\mathrm{ver}} = C^b_{ab}\frac{\partial}{\partial P_a} \in \Gamma^\infty(T(T^*G))$. Following [7] we consider the formal series of differential operators

$$N = \exp\left(\frac{\lambda}{2\mathrm{i}}\left(\Delta_0 - \Delta^{\mathrm{ver}}\right)\right) \tag{141}$$

acting on $\mathcal{C}^\infty(T^*G)[[\lambda]]$. A non-trivial integration by parts (even possible for arbitrary cotangent bundles [7]) yields then the result

$$\int_G \overline{\phi}\, \varrho_{\mathrm{Std}}(f)\psi\, \mathrm{d}^{\mathrm{left}}\, g = \int_G \overline{\varrho_{\mathrm{Std}}(N^2\overline{f})\phi}\, \psi\, \mathrm{d}^{\mathrm{left}}\, g \tag{142}$$

for $\phi, \psi \in \mathcal{C}^\infty_0(G)[[\lambda]]$. From this failure of ϱ_{Std} being compatible with complex conjugation we see that the definition

$$f \star_{\mathrm{G}} g = N^{-1}(Nf \star_{\mathrm{Std}} Ng) \tag{143}$$

yields again a bidifferential star product for which

$$\varrho_{\mathrm{Weyl}}(f)\psi = \varrho_{\mathrm{Std}}(Nf)\psi = \iota^*\left(Nf \star_{\mathrm{Std}} \mathrm{pr}^*\,\phi\right) \tag{144}$$

defines a *-representation on $\mathcal{C}^\infty_0(G)[[\lambda]]$. This is the canonical Hermitian star product on G, originally constructed in [18]: there, $\star_{\mathrm{G}}$ was obtained from the observation that $\mathrm{Pol}^\bullet(T^*G)^G \cong \mathrm{Pol}^\bullet(\mathfrak{g}^*) \cong \mathrm{S}^\bullet(\mathfrak{g})$, using the PBW isomorphism to the universal enveloping algebra of $\mathfrak{g}$, and pulling back the product. In fact, $\star_{\mathrm{G}}$ turns out to be strongly invariant and $\mathrm{Pol}^\bullet(T^*G)^G[[\lambda]]$ forms a subalgebra being isomorphic to the "formal" universal enveloping algebra. The representation ϱ_{Weyl} is also called the Schrödinger representation in Weyl ordering since for the Lie group $G = \mathbb{R}^n$ this indeed reproduces the usual canonical quantization in Weyl ordering.

7.3 The bimodule structure on $\mathcal{C}^\infty_{\mathrm{cf}}(M_{\mathrm{red}} \times G)[[\lambda]]$

We will now use the strongly invariant star product $\star = \star_{\mathrm{red}} \otimes \star_{\mathrm{G}}$ on $M = M_{\mathrm{red}} \times T^*G$. For the quantized Koszul operator we will have the following result:

Lemma 7.2 *Let $x \in \mathcal{C}^\infty(M, \Lambda^1_{\mathbb{C}}\mathfrak{g})[[\lambda]]$. Then we have*

$$\iota^* N\boldsymbol{\partial} x = 0. \tag{145}$$

Proof. First it is clear that the M_{red}-components do not enter at all. Thus we can compute the left hand side of (145) on T^*G alone. We have $\Delta_0 J_a = 0$ and $\Delta^{\mathrm{ver}} J_a = -C^b_{ab}$ by the explicit form for Δ^{ver} and $J_a = -P_a$. Thus $NJ_a = J_a - \frac{\mathrm{i}\lambda}{2}C^b_{ab}$. Using $\boldsymbol{\partial} x = x^a \star_{\mathrm{G}} J_a + \frac{\mathrm{i}\lambda}{2}C^b_{ab}x^a$ according to (31) we get

$$N\boldsymbol{\partial} x = (Nx^a) \star_{\mathrm{Std}} (NJ_a) + \frac{\mathrm{i}\lambda}{2}C^b_{ab}Nx^a = (Nx^a) \star_{\mathrm{Std}} J_a.$$

Note that at this point our choice $\kappa = \frac{1}{2}$ in (31) enters again. Now ϱ_{Std} is a symbol calculus where $f \in \mathrm{Pol}^\bullet(T^*G)$ corresponds to a differential operator $\varrho_{\mathrm{Std}}(f)$ with $\varrho_{\mathrm{Std}}(f)1 = 0$ iff f has no contributions from polynomial degree 0. This means that $(Nx^a) \star_{\mathrm{Std}} J_a$ is at least linear in the momenta, no matter what $x^a \in \mathcal{C}^\infty(T^*G)[[\lambda]]$ is. Thus (145) follows. $\square$

Corollary 7.3 *For the deformed restriction map $\boldsymbol{\iota}^*$ we have*

$$\boldsymbol{\iota}^* = \iota^* \circ \left(\mathsf{id} + (\boldsymbol{\partial}_1 - \partial_1)\, h_0\right)^{-1} = \iota^* \circ N. \tag{146}$$

Proof. By the general argument from [6, Prop. 25] we know that the deformed restriction map $\boldsymbol{\iota}^*$ is uniquely characterized by the following three properties: its classical limit is ι^*, $\boldsymbol{\iota}^*\boldsymbol{\partial}_1 = 0$ and $\boldsymbol{\iota}^* \operatorname{prol} = \mathsf{id}$. Clearly, $\iota^* \circ N$ fulfills the first requirement. Also the last requirement is clear as the exponent of N differentiates in momenta direction and hence vanishes on pull-backs $\operatorname{pr}^* \phi$. Finally, the second requirements is fulfilled by Lemma 7.2. □

Thus we have computed the formal series of differential operators from Lemma 3.5 explicitly in this situation. Of course, handling a formal series of differential operators like N is much easier that the non-local operator $(\mathsf{id} + (\boldsymbol{\partial}_1 - \partial 1))^{-1}$. We arrive at the following statement:

Theorem 7.4 *Let $f \in \mathcal{C}^\infty(M_{\mathrm{red}} \times T^*G)[[\lambda]]$, $\phi \in \mathcal{C}^\infty(M_{\mathrm{red}} \times G)[[\lambda]]$, and $u, v \in \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ be given.*

i.) The left module structure (42) is explicitly given by

$$f \bullet \phi = \iota^* \left(N f (\star_{\mathrm{red}} \otimes \star_{\mathrm{Std}}) \operatorname{prol}(\phi) \right) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\lambda}{\mathrm{i}} \right)^r \iota^* \left(\frac{\partial^r f}{\partial P_{a_1} \cdots \partial P_{a_r}} \right) \star_{\mathrm{red}} \mathscr{L}_{X_{a_1}} \cdots \mathscr{L}_{X_{a_r}} \phi, \tag{147}$$

where $\star_{\mathrm{red}}$ is extended to $M_{\mathrm{red}} \times G$ as usual.

ii.) The reduced star product (52) reproduces $\star_{\mathrm{red}}$ on M_{red}. The right module structure (53) is explicitly given by

$$\phi \bullet_{\mathrm{red}} u = \phi \star_{\mathrm{red}} \pi^* u. \tag{148}$$

iii.) The inner product (97) is explicitly given by

$$\langle \phi, \psi \rangle_{\mathrm{red}} (p) = \int_G \left(\overline{\phi} \star_{\mathrm{red}} \psi \right) (p, g) \, \mathrm{d}^{\mathrm{left}} g, \tag{149}$$

where $p \in M_{\mathrm{red}}$ and $\phi, \psi \in \mathcal{C}^\infty_{\mathrm{cf}}(M_{\mathrm{red}} \times G)[[\lambda]]$.

Proof. For the first part we compute

$$\begin{aligned} f \bullet \phi &= \boldsymbol{\iota}^*(f \star \operatorname{prol}(\phi)) \\ &= \iota^* N (f (\star_{\mathrm{red}} \otimes \star_{\mathrm{G}}) \operatorname{prol}(\phi)) \\ &= \iota^* \left(N f (\star_{\mathrm{red}} \otimes \star_{\mathrm{Std}}) \operatorname{prol}(\phi) \right) \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\lambda}{\mathrm{i}} \right)^r \iota^* \left(\frac{\partial^r f}{\partial P_{a_1} \cdots \partial P_{a_r}} \right) \star_{\mathrm{red}} \mathscr{L}_{X_{a_1}} \cdots \mathscr{L}_{X_{a_r}} \phi. \end{aligned}$$

The second part is clear since $\operatorname{prol}(\pi^* u) \star \operatorname{prol}(\pi^* v) = \operatorname{prol}(\pi^*)(\star_{\mathrm{red}} \otimes \star_{\mathrm{G}}) \operatorname{prol}(\pi^* v) = \operatorname{prol}(\pi^*(u \star_{\mathrm{red}} v))$. Indeed, the standard-ordered product as well as the canonical star product reduce to the pointwise product if both functions are independent of the momenta. A further application of N yields nothing new for the same reason showing that our general construction reproduces $\star_{\mathrm{red}}$ as expected. For the right module structure we can argue similarly. Finally, for the third part we have

$$\begin{aligned} \pi^* \langle \phi, \psi \rangle_{\mathrm{red}} &= \int_G \mathsf{L}^*_{g^{-1}} \boldsymbol{\iota}^* \left(\overline{\operatorname{prol}(\phi)} \star \operatorname{prol}(\psi) \right) \mathrm{d}^{\mathrm{left}} g \\ &= \int_G \mathsf{L}^*_{g^{-1}} \iota^* N \left(\operatorname{prol} \left(\overline{\phi} \star_{\mathrm{red}} \psi \right) \right) \mathrm{d}^{\mathrm{left}} g \\ &= \int_G \mathsf{L}^*_{g^{-1}} \left(\overline{\phi} \star_{\mathrm{red}} \psi \right) \mathrm{d}^{\mathrm{left}} g, \end{aligned}$$

since again N and $\star_{\mathrm{G}}$ act trivially on functions not depending on the momenta. □

7.4 Rieffel induction

Having an explicit description of the bimodule structure and the inner product we can compute the result of the corresponding Rieffel induction as well.

To simplify things slightly, we will restrict to the following unital *-subalgebra $(\mathcal{C}^\infty(M_{\mathrm{red}}) \otimes_{\mathbb{C}} \mathcal{C}^\infty(T^*G))[[\lambda]]$ of $\mathcal{C}^\infty(M_{\mathrm{red}} \times T^*G)[[\lambda]]$. Thanks to the factorization of the star product, this is indeed a subalgebra. Moreover, it acts on $(\mathcal{C}^\infty(M_{\mathrm{red}}) \otimes_{\mathbb{C}} \mathcal{C}^\infty_0(G))[[\lambda]]$ of $\mathcal{C}^\infty_{\mathrm{cf}}(M_{\mathrm{red}} \times G)[[\lambda]]$ which becomes a bimodule for $(\mathcal{C}^\infty(M_{\mathrm{red}}) \otimes_{\mathbb{C}} \mathcal{C}^\infty(T^*G))[[\lambda]]$ from the left and $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ from the right as before. Clearly, all our previous results restrict well to this situation. Note that the $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$-valued inner product is still *full* when restricted to $(\mathcal{C}^\infty(M_{\mathrm{red}}) \otimes_{\mathbb{C}} \mathcal{C}^\infty_0(G))[[\lambda]]$. The reason why we restrict to this subalgebra and this submodule is that the Rieffel induction functor will have a very nice end explicit form here.

Remark 7.5 The natural locally convex topologies of smooth functions (with compact support) make $\mathcal{C}^\infty(M_{\mathrm{red}}) \otimes_{\mathbb{C}} \mathcal{C}^\infty_0(G)$ a dense subspace of $\mathcal{C}^\infty_{\mathrm{cf}}(M_{\mathrm{red}} \times G)$ and similarly for $\mathcal{C}^\infty(M_{\mathrm{red}}) \otimes_{\mathbb{C}} \mathcal{C}^\infty(T^*G)$. Thus, morally, the above restriction is not severe: as soon as one enters a more topological framework all the (hopefully continuous) structure maps should be determined by their behaviour on these dense subspaces. Of course, the λ-adic topology does not fit together well with the smooth function topology, at least in a naive way. Nevertheless, we consider this to be a technicality which may only cause artificial difficulties but no conceptual ones.

The above simplification allows to re-interpret the factorizing case in the following, purely algebraic way. Assume that $\mathcal{A}_{\mathrm{red}}$ and $\mathcal{B}$ are unital *-algebras over C and $\mathcal{A} = \mathcal{B} \otimes_{\mathsf{C}} \mathcal{A}_{\mathrm{red}}$ is their algebraic tensor product, again endowed with its canonical unital *-algebra structure. Assume moreover, that $\mathcal{B}_0 \subseteq \mathcal{B}$ is a *-ideal and

$$\omega : \mathcal{B}_0 \longrightarrow \mathsf{C} \tag{150}$$

is a positive linear functional with Gel'fand ideal $\mathcal{J}_\omega \subseteq \mathcal{B}_0$. Then it is well-known that the GNS representation of $\mathcal{B}_0$ on $\mathcal{B}_0/\mathcal{J}_\omega$ extends to a *-representation of $\mathcal{B}$ on $\mathcal{B}_0/\mathcal{J}_\omega$ in the canonical way.

Remark 7.6 In our example we have $\mathcal{A}_{\mathrm{red}} = \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ with $\star_{\mathrm{red}}$ and $\mathcal{B} = \mathcal{C}^\infty(T^*G)[[\lambda]]$ as well as $\mathcal{B}_0 = \mathcal{C}^\infty_0(T^*G)[[\lambda]]$. The positive functional ω is then the Schrödinger functional

$$\omega(f) = \int_G \iota^* f \, \mathrm{d}^{\mathrm{left}} g \stackrel{(*)}{=} \langle 1, \varrho_{\mathrm{Weyl}}(f) 1 \rangle \,, \tag{151}$$

see [27, Prop. 7.1.35] for the justification of $(*)$. Moreover, one knows that the GNS representation corresponding to ω reproduces the Schrödinger representation ϱ_{Weyl} on $\mathcal{C}^\infty_0(G)[[\lambda]]$ with the usual L^2-inner product, see e.g. [27, Satz 7.2.26] for a discussion and further references.

We will now make use of the external tensor product of pre Hilbert modules [11, Sect. 4]: for two *-algebras $\mathcal{A}_i$ with $i = 1, 2$ and corresponding pre Hilbert right $\mathcal{A}_i$-modules $\mathcal{E}_i$ one defines on $\mathcal{E}_1 \otimes \mathcal{E}_2$ an inner product by

$$\left\langle x \otimes x', y \otimes y' \right\rangle^{\mathcal{E}_1 \otimes \mathcal{E}_2}_{\mathcal{A}_1 \otimes \mathcal{A}_2} = \langle x, y \rangle^{\mathcal{E}_1}_{\mathcal{A}_1} \otimes \left\langle x', y' \right\rangle^{\mathcal{E}_2}_{\mathcal{A}_2} \tag{152}$$

with values in the *-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$. It turns out that (152) is again completely positive once both inner products $\langle \cdot, \cdot \rangle^{\mathcal{E}_i}_{\mathcal{A}_i}$ were completely positive, see [11, Remark 4.12]. However, it might happen that (152) is degenerate. Thus the external tensor product is defined analogously to the internal tensor product (108) as the quotient

$$\mathcal{E}_1 \otimes_{\mathrm{ext}} \mathcal{E}_2 = \mathcal{E}_1 \otimes \mathcal{E}_2 \big/ \left(\mathcal{E}_1 \otimes \mathcal{E}_2\right)^\perp \tag{153}$$

in order to get again a non-degenerate inner product. Needless to say, the construction of $\otimes_{\mathsf{ext}}$ is functorial in a good sense similarly to the internal tensor product.

Now we take $\mathcal{A}_{\mathrm{red}}$ as a right $\mathcal{A}_{\mathrm{red}}$-module with its canonical completely positive inner product $\langle a, a' \rangle = a^* a'$. Then we can form the external tensor product with $\mathcal{B}_0/\mathcal{J}_\omega$ endowed with its pre Hilbert space structure. Thus we consider

$$\mathcal{E} = \left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes_{\mathsf{ext}} \mathcal{A}_{\mathrm{red}}. \tag{154}$$

The completely positive inner product (152) becomes on factorizing representatives in $\left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes \mathcal{A}_{\mathrm{red}}$

$$\left\langle [b] \otimes a, [b'] \otimes a' \right\rangle_\omega = \omega(b^* b') a^* a', \tag{155}$$

where $b, b' \in \mathcal{B}_0$ and $a, a' \in \mathcal{A}_{\mathrm{red}}$. Typically, the degeneracy space of (155) will be trivial already whence the quotient (153) is unnecessary.

Lemma 7.7 *Let $\mathcal{A}_{\mathrm{red}}$ and $\mathcal{B}$ be unital *-algebras, $\mathcal{B}_0 \subseteq \mathcal{B}$ a *-ideal, $\omega : \mathcal{B}_0 \longrightarrow \mathsf{C}$ a positive linear functional with Gel'fand ideal $\mathcal{J}_\omega$, and $\mathcal{E} = \left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes_{\mathsf{ext}} \mathcal{A}_{\mathrm{red}}$.*

- *i.) The pre Hilbert right $\mathcal{A}_{\mathrm{red}}$-module $\mathcal{E}$ carries a *-representation of $\mathcal{A} = \mathcal{B} \otimes \mathcal{A}_{\mathrm{red}}$ coming from the canonical $\mathcal{A}$-left module structure on $\left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes \mathcal{A}_{\mathrm{red}}$.*
- *ii.) If $\mathrm{im}(\omega\big|_{\mathcal{B}_0 \cdot \mathcal{B}_0}) = \mathsf{C}$ then (155) is full.*

Proof. The first statement is part of the functoriality of the external tensor product and in fact easy to verify. The second part is clear. □

Remark 7.8 In our example, after the usual identification, we have

$$\mathcal{E} = \mathcal{C}_0^\infty(G)[[\lambda]] \otimes_{\mathsf{ext}} \mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]] \subseteq \left(\mathcal{C}_0^\infty(G) \otimes \mathcal{C}^\infty(M_{\mathrm{red}})\right)[[\lambda]], \tag{156}$$

with the inner product being precisely $\langle \cdot, \cdot \rangle_{\mathrm{red}}$ from (149). In fact, $\mathcal{E}$ as constructed in (154) needs not to be λ-adically complete in general but it will be dense in the right hand side of (156). Note that ω fulfills the hypothesis of Lemma 7.7, *ii.)*.

We can now use the bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_{\mathrm{red}}}$ to induce *-representations. Thus let $\mathcal{D}$ be an auxiliary *-algebra over C for the coefficients.

Proposition 7.9 *We have a natural equivalence*

$$\mathsf{R}_{\mathcal{E}}(\cdot) \cong \left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes_{\mathsf{ext}} \cdot : {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{A}_{\mathrm{red}}) \longrightarrow {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{A}). \tag{157}$$

Proof. On objects, i.e. on a strongly non-degenerate *-representation ${}_{\mathcal{A}_{\mathrm{red}}}\mathcal{H}_{\mathcal{D}} \in {}^*\text{-}\mathsf{Rep}_{\mathcal{D}}(\mathcal{A}_{\mathrm{red}})$ the Rieffel induction is given by

$${}_{\mathcal{A}}\mathcal{E}_{\mathcal{A}_{\mathrm{red}}} \widehat{\otimes}_{\mathcal{A}_{\mathrm{red}}} {}_{\mathcal{A}_{\mathrm{red}}}\mathcal{H}_{\mathcal{D}} = \left(\left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes_{\mathsf{ext}} \mathcal{A}_{\mathrm{red}}\right) \widehat{\otimes}_{\mathcal{A}_{\mathrm{red}}} {}_{\mathcal{A}_{\mathrm{red}}}\mathcal{H}_{\mathcal{D}}\,.$$

This motivates to use the "associativity" of the tensor product to implement the natural equivalence (157). Due to the presence of the two quotient procedures in $\otimes_{\mathsf{ext}}$ and $\widehat{\otimes}_{\mathcal{A}_{\mathrm{red}}}$ we have to be slightly careful. Nevertheless, the C-linear map defined by

$$\mathsf{a} : \left(\left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes \mathcal{A}_{\mathrm{red}}\right) \otimes_{\mathcal{A}_{\mathrm{red}}} {}_{\mathcal{A}_{\mathrm{red}}}\mathcal{H}_{\mathcal{D}} \ni ([b] \otimes a) \otimes \phi \;\mapsto\; [b] \otimes (a \cdot \phi) \in \left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes {}_{\mathcal{A}_{\mathrm{red}}}\mathcal{H}_{\mathcal{D}} \tag{$*$}$$

turns out to be isometric with respect to the inner products on both sides. Thus it passes to the quotients and yields an isometric and now injective map

$$\mathsf{a} : \left(\left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes_{\mathsf{ext}} \mathcal{A}_{\mathrm{red}}\right) \widehat{\otimes}_{\mathcal{A}_{\mathrm{red}}} {}_{\mathcal{A}_{\mathrm{red}}}\mathcal{H}_{\mathcal{D}} \longrightarrow \left(\mathcal{B}_0/\mathcal{J}_\omega\right) \otimes_{\mathsf{ext}} {}_{\mathcal{A}_{\mathrm{red}}}\mathcal{H}_{\mathcal{D}}\,. \tag{$**$}$$

Since $\mathcal{A}_{\mathrm{red}}$ is unital and $\mathbb{1}_{\mathcal{A}_{\mathrm{red}}} \cdot \phi = \phi$ for all $\phi \in {}_{\mathcal{A}_{\mathrm{red}}}\mathcal{H}_{\mathcal{D}}$ by assumption, we see that ($*$) and hence also ($**$) is surjective. Thus a is unitary. It is now easy to check that a is compatible with intertwiners and hence natural as claimed. □

Remark 7.10 From this proposition we arrive at the following picture for our example: up to the completion issues the Rieffel induction with $\mathcal{C}^\infty_{\mathrm{cf}}(M_{\mathrm{red}} \times G)[[\lambda]]$ simply consists in tensoring the given *-representation of $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ with the Schrödinger representation (144) on $\mathcal{C}^\infty_0(G)[[\lambda]]$. Note that once the *-representation of $\mathcal{C}^\infty(M_{\mathrm{red}})[[\lambda]]$ is specified we have everywhere very explicit formulas.

A Densities on principal bundles

In this appendix we collect some well-known basic facts on densities on a principal bundle. The principal bundle will be denoted by $\pi : C \circlearrowleft G \longrightarrow M_{\mathrm{red}}$ as before. We follow the tradition that the group acts from the right, g acting via $\mathsf{R}_g : C \longrightarrow C$. The corresponding left action, as we used it throughout the main text, is then $\mathsf{L}_g = \mathsf{R}_{g^{-1}}$.

We fix once and for all a normalization of the constant positive density $|\,\mathrm{d}^N x|$ on the vector space $\mathfrak{g}$. Moreover, we consider a *horizontal lift*

$$^{\mathrm{hor}} : \Gamma^\infty(TM_{\mathrm{red}}) \longrightarrow \Gamma^\infty(TC), \tag{158}$$

which can e.g. be obtained from a principal connection. For a density $\Omega \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M_{\mathrm{red}})$ we can define a new density $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*C)$ as follows: for $c \in C$ we choose a basis $X_1, \ldots, X_n \in T_{\pi(c)}M_{\mathrm{red}}$ and define

$$\mu_c\left(X_1^{\mathrm{hor}}(c), \ldots, X_n^{\mathrm{hor}}(c), (e_1)_C, \ldots, (e_N)_C\right) = \Omega_{\pi(c)}(X_1, \ldots, X_n)|\,\mathrm{d}^N x|(e_1, \ldots, e_N). \tag{159}$$

This yields indeed a smooth density μ on C which has the following properties:

Proposition A.1 *Let $C \circlearrowleft G \longrightarrow M_{\mathrm{red}}$ be a principal bundle.*

i.) The definition (159) *yields a smooth, well-defined density $\mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*C)$ which is independent on the choice of the horizontal lift.*

ii.) Let $c \in C$ then μ_c is positive iff $\Omega_{\pi(c)}$ is positive.

iii.) For all $g \in G$ one has

$$\mathsf{R}_g^*\mu = \frac{1}{\Delta(g)}\mu. \tag{160}$$

iv.) The map

$$\Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*M_{\mathrm{red}}) \ni \Omega \;\mapsto\; \mu \in \Gamma^\infty(|\Lambda^{\mathrm{top}}|T^*C) \tag{161}$$

is a $\mathcal{C}^\infty(M_{\mathrm{red}})$-module monomorphism which is surjective onto those densities satisfying (160).

Proof. The first part is a simple verification that μ transforms correctly under a change of the bases. Moreover, since passing to another horizontal lift changes X^{hor} by vertical terms, it follows from this block-structure that μ does not depend on the choice of the horizontal lift. The second part is clear. For the third, note that the fundamental vector field ξ_C satisfies $\mathsf{R}_g^*\xi_C = (\mathrm{Ad}_g\,\xi)_C$. Then (160) follows easily as we can choose an invariant horizontal lift, i.e. we have $\mathsf{R}_g^*X^{\mathrm{hor}} = X^{\mathrm{hor}}$ for all vector fields $X \in \Gamma^\infty(TM)$. Finally, (161) is clearly $\mathcal{C}^\infty(M_{\mathrm{red}})$-linear (along π^*) and injective. Now chose $\Omega > 0$ and thus $\mu > 0$. If $\tilde{\mu}$ is a density with (160) then $\tilde{\mu} = \pi^*u\mu$ with some $u \in \mathcal{C}^\infty(M_{\mathrm{red}})$ showing the surjectivity. □

We need some local expressions for μ in order to compute integrations with respect to μ. Thus let $U \subseteq M_{\mathrm{red}}$ be a small enough open subset such that there exists a G-equivariant diffeomorphism

$$\Phi : U \times G \longrightarrow \pi^{-1}(U) \subseteq C, \tag{162}$$

i.e. a trivialization. Since we trivialize C as a *right* principal bundle, the fundamental vector field $\xi_{U\times G}$ on $U \times G$ at (p, g) is simply given by minus the *left* invariant vector field $\xi_{U\times G} = -T_e\mathsf{l}_g(\xi)$.

Note that the minus sign appears as we define the fundamental vector fields with respect to the left action. However, in the density $|\,\mathrm{d}^n x|$ this does not matter anyway. Since we are free to choose the horizontal lift we take $X^{\mathrm{hor}}(p, g) = X(p)$ for $X \in \Gamma^\infty(TU)$. Then the definition of μ just gives

$$\mu = \Omega \boxtimes \mathrm{d}^{\mathrm{left}}\, g, \tag{163}$$

i.e. the (external) tensor product of the *left* invariant Haar density and Ω. Thus for $\phi \in \mathcal{C}_0^\infty(C)$ with support in $\pi^{-1}(U)$ we get

$$\int_C \phi\, \mu = \int_{U\times G} \phi(p, g)\Omega(p)\, \mathrm{d}^{\mathrm{left}}\, g. \tag{164}$$

A more global interpretation of this local formula is obtained as follows: for $\phi \in \mathcal{C}_0^\infty(C)$ the integral

$$\int_G \mathsf{R}_g^*\phi\, \mathrm{d}^{\mathrm{left}}\, g\Big|_c = \int_G \phi(\mathsf{R}_g(c))\, \mathrm{d}^{\mathrm{left}}\, g \tag{165}$$

yields an invariant smooth function on C since $\mathrm{d}^{\mathrm{left}}\, g$ is left invariant. Thus it is of the form $\pi^* u$ with some function $u \in \mathcal{C}_0^\infty(M_{\mathrm{red}})$. Note that u still has compact support. Using the above local result and a partition of unity argument we see that for this function u we have

$$\int_{M_{\mathrm{red}}} u\, \Omega = \int_C \phi\mu. \tag{166}$$

With some slight abuse of notation (omitting the π^*) we therefor write

$$\int_C \phi\, \mu = \int_{M_{\mathrm{red}}} \left(\int_G \mathsf{R}_g^*\phi\, \mathrm{d}^{\mathrm{left}}\, g\right)\Omega. \tag{167}$$

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