Upper-lower class tests for weighted i.i.d. sequences and martingales

István Berkes, Siegfried Hörmann and Michel Weber

(Accepted for publication in Journal of Theoretical Probability.)

Abstract

After earlier work of Lévy, Kolmogorov, Erdős and Petrovski, the upper-lower class behavior of partial sums of independent random variables was described completely in two seminal papers of Feller (1943, 1946). Feller proved that for individually bounded random variables X_n, the form of the upper-lower class test depends sensitively on the bounds of X_n and he also found the precise moment condition for Kolmogorov's integral test in the case of i.i.d. random variables X_n. In this paper we investigate the upper-lower class behavior of weighted i.i.d. sums $\sum_{k=1}^n a_k X_k$ where X_k satisfy Feller's sharp moment condition. In contrast to Feller's results, we show that the refined LIL behavior of such sums depends not on the growth properties of (a_n), but on its *arithmetical* distribution, permitting pathological behavior even for uniformly bounded (a_n). We prove analogous results for weighted sums of stationary martingale difference sequences. These are new even in the unweighted case and complement the sharp results of Einmahl and Mason obtained in the bounded case. Finally, we prove a general upper-lower class test for unbounded martingales improving several earlier results in the literature.

MSC 2000 Subject Classification. 60F15, 60G50, 60G42

Keywords. Weighted i.i.d. sequences, upper-lower class tests, law of the iterated logarithm, martingales

[0]István Berkes, Institute of Statistics, Graz University of Technology, Steyrergasse 17/IV, A-8010 Graz, Austria, Email: berkes@tugraz.at

Siegfried Hörmann, Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA, Email: hormann@math.utah.edu

Michel Weber, Mathématique (IRMA), Université Louis-Pasteur et C.N.R.S. 7 rue René Descartes, 67084 Strasbourg Cedex, France E-mail: weber@math.u-strasbg.fr

1 Introduction and results

Let $X_1, X_2, \ldots$ be independent random variables with mean 0 and finite variances and let $S_n = \sum_{k=1}^n X_k$, $s_n^2 = \sum_{k=1}^n EX_k^2$. By Kolmogorov's law of the iterated logarithm (see [14]), if

$$|X_n| \le \varepsilon_n s_n / (\log\log s_n^2)^{1/2} \tag{1.1}$$

with a positive numerical sequence $\varepsilon_n \to 0$, then

$$\limsup_{n\to\infty} (2s_n^2 \log\log s_n^2)^{-1/2} S_n = 1 \qquad \text{a.s.} \tag{1.2}$$

Here, and in the sequel, $\log\log x$ is meant as $\log\log(x \vee e^e)$. Condition (1.1) is best possible: assuming only

$$|X_n| \le K s_n / (\log\log s_n^2)^{1/2}$$

with some constant $K > 0$, relation (1.2) is generally false (see Marczinkiewicz and Zygmund [16], Weiss [20]). A much more refined result was proved by Feller [7] , who showed that if

$$|X_n| \le K s_n / (\log\log s_n^2)^{3/2} \tag{1.3}$$

and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function, then

$$P\{S_n > s_n \varphi(s_n^2) \quad \text{i.o.}\} = 0 \quad \text{or} \quad 1 \tag{1.4}$$

according as

$$I(\varphi) := \int_1^\infty t^{-1} \varphi(t) e^{-\varphi(t)^2/2} dt < \infty \quad \text{or} \quad = \infty. \tag{1.5}$$

As customary, we say that φ belongs to upper or lower class with respect to S_n according as the probability in (1.4) is 0 or 1. Condition (1.3) is also best possible: replacing it by

$$|X_n| \le K_n s_n / (\log\log s_n^2)^{3/2} \tag{1.6}$$

with any fixed sequence $K_n \to \infty$, the test (1.4)–(1.5) becomes generally false.

Using truncation, the above results can be easily extended to unbounded r.v.'s X_n, but no sharp condition for the upper-lower class test (1.4)–(1.5) has been found in the unbounded case, except for i.i.d. sequences (X_n) where Feller [8] proved that $EX_1 = 0$, $EX_1^2 = 1$ and

$$EX_1^2 I(|X_1| > t) = O((\log\log t)^{-1}) \tag{1.7}$$

imply (1.4)–(1.5) and the last condition is best possible. In particular, the test holds if

$$EX_1^2 \log\log |X_1| < \infty. \tag{1.8}$$

This condition is also optimal in the sense that given any function $\psi(x) = o(x^2 \log\log x)$, $x \to \infty$, there is an i.i.d. sequence (X_n) with $EX_1 = 0$, $EX_1^2 = 1$ and $E\psi(|X_1|) < +\infty$ such that the test (1.4)–(1.5) fails. Note that Feller's proof in [?] contains a gap and is correct only for symmetric X_1. Einmahl [3] was the first to give a complete proof.

The purpose of the present paper is to study the upper-lower class behavior of weighted sums $S_n = \sum_{k=1}^n a_k X_k$, where (X_n) is an i.i.d. sequence satisfying $EX_1 = 0$ and $EX_1^2 = 1$. We will assume (1.8) to guarantee that the test (1.4)–(1.5) is satisfied for $a_k = 1$. As a first orientation, consider the analogous problem in the bounded case, i.e. for the sums $S_n = \sum_{k=1}^n a_k X_k$, where (X_n) are independent, zero mean r.v.'s satisfying (1.3). This problem was completely solved by Feller [6, Theorem 5] who showed that if

$$a_n = O((\log\log s_n^2)^\alpha) \quad 0 < \alpha < 1,$$

then the test (1.4)–(1.5) remains valid, but the exponent $-\varphi(t)^2/2$ in the integral (1.5) should be replaced by a polynomial of $\varphi(t)$ of degree $l+2$ where l is the smallest positive integer with $(l-1)/l < \alpha$. For example, if $0 < \alpha \le 1/2$, then in the exponent of (1.5) an extra term $c_1\varphi(t)^3$ appears, if $1/2 < \alpha \le 2/3$, then in (1.5) the extra terms $c_1\varphi(t)^3 + c_2\varphi(t)^4$ appear, etc. If α approaches 1 (which means approaching the Kolmogorov condition (1.1)) then the number of terms in the exponent of (1.5) becomes infinite and in the limiting case even the LIL breaks down.

In analogy with the above result, one could expect that the upper-lower class behavior of $S_n = \sum_{k=1}^n a_k X_k$, for an i.i.d. sequence X_n is determined by the growth speed of (a_n). As we will see, however, this is not the case: the fine asymptotics of $S_n = \sum_{k=1}^n a_k X_k$ depends not on the speed of growth of (a_n), but its *arithmetical* distribution. For example, we will see that there exists a bounded sequence (a_n) such that the test (1.4)–(1.5) fails for $a_n X_n$. Our main result is:

Theorem 1. *Let (a_k) be a sequence of nonzero real numbers, $s_n^2 = \sum_{k=1}^n a_k^2$ and assume that $s_n \to \infty$, $|a_n| = O(s_n^{1-\delta})$ for some $\delta > 0$. Let $M(x) = \#\{k \ge 1 : |s_k/a_k| \le x\}$ for $x > 0$ and assume*

$$M(x) \ll x^2 \quad \text{as } x \to \infty. \tag{1.9}$$

Then for any i.i.d. sequence (X_n) satisfying $EX_1 = 0$, $EX_1^2 = 1$ and (1.8), the weighted partial sums $S_n = \sum_{k=1}^n a_k X_k$ satisfy the test (1.4)–(1.5). Conversely, if the last statement is valid, *we have*

$$M(x) \ll x^2(\log\log x)^2 \quad \textit{as } x \to \infty. \tag{1.10}$$

Here, and in the sequel, $\ll$ means the same as the O notation. A necessary and sufficient condition for the weighted strong law of large numbers in terms of the distributional properties of the coefficients was given by Jamison et al. [13]; a similar criterion for the weighted LIL is implicit in Fisher [9]. (See also Weber and Lin [19].) Adapting an argument of Jamison et al. [13] it is easy to show that there exists a uniformly bounded sequence (a_k) with

$$\limsup_{x\to\infty} M(x)/(x^2 \log x) \geq 1/2.$$

Thus by the second half of Theorem 1, there exists an i.i.d. sequence X_n satisfying $EX_1 = 0$, $EX_1^2 = 1$ and (1.8) such that $a_n X_n$ fails the test (1.4)–(1.5) for some uniformly bounded sequence (a_n) with $s_n^2 = \sum_{k=1}^n a_k^2 \to \infty$.

It is important to note that in Theorem 1 we work under the minimal moment condition (1.8). If we assume the stronger condition $EX_1^2(\log X_1)^\gamma < \infty$ for some $\gamma > 0$, then it follows easily that $S_n = \sum_{k=1}^n a_k X_k$ satisfies the integral test (1.4)–(1.5) provided $|a_n| = O(s_n^{1-\delta})$ for some $\delta > 0$, i.e. the arithmetic condition (1.9) in Theorem 1 becomes unnecessary.

In what follows, we give examples for irregular unbounded sequences (a_n) satisfying the arithmetic condition (1.9). A good source of such examples is number theory, where the irregularity of the sequence of primes provides the required properties of (a_n). A function f defined on positive integers is called *strongly additive* if $f(mn) = f(m) + f(n)$ provided m and n are coprimes and $f(p^r) = f(p)$ for p prime and $r = 1, 2, \ldots$. Clearly, $f(n) = \sum_{p|n} f(p)$ (here and in the sequel p denotes primes), which causes f to behave rather irregularly.

Example 1. *Let $f \geq 0$ be a strongly additive number-theoretic function with $D_N^2 := \sum_{p\leq N} \frac{f^2(p)}{p} \to +\infty$ and assume $f(p) = o(D_p)$, $p \to \infty$. Then the sequence $a_n = f(n)$ satisfies condition (1.9).*

For example, we can choose $f(n) = \omega(n)$, the number of different prime factors of n. Additive functions f satisfying the assumptions of Example 1 play a prominent role in probabilistic number theory, namely the central limit theory for additive functions, see Erdős and Kac [6] and Kubilius [15].

Another simple example for a suitable weight sequence in Theorem 1 is provided by the realizations of stationary ergodic sequences.

Example 2. *Let $\{\xi_n,\, n \geq 1\}$ be a stationary ergodic sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ satisfying $0 < E|\xi_1|^p < \infty$ for some $p > 2$. Then for almost every $\omega \in \Omega$, the sequence $a_n = \xi_n(\omega)$ satisfies condition (1.9).*

For example, let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function with period 1 satisfying $0 < \int_0^1 |f(x)|^p dx < \infty$ for some $p > 2$. Then $a_n = f(2^n x)$ satisfies (1.9) for almost every x in the sense of the Lebesgue measure.

Our next theorem extends Theorem 1 for weighted sums of stationary martingale difference sequences. Let $\{X_n, n \in \mathbb{Z}\}$ be a strictly stationary ergodic sequence satisfying $E[X_n|\mathcal{F}_{n-1}] = 0$, $EX_0^2 = 1$ where $\mathcal{F}_n = \sigma(X_j,\, j \leq n)$. Let (a_n) be a sequence of nonzero real numbers, put $s_n^2 = \sum_{k=1}^n a_k^2 E(X_k^2|\mathcal{F}_{k-1})$ and define $M(x)$ as in Theorem 1. To keep the arithmetical condition $M(x) \ll x^2$ nonrandom, we will assume that $s_n^2 \sim B_n^2$ a.s. with $B_n^2 = \sum_{k=1}^n a_k^2$, or, alternatively,

$$B_n^{-2} \sum_{k=1}^n a_k^2 X_k^2 \longrightarrow 1 \qquad \text{a.s.} \tag{1.11}$$

For $a_k = 1$ the validity of (1.11) is immediate from the ergodic theorem and in the case when (X_n) is an i.i.d. sequence, a classical theorem of Jamison et al. [13] provides necessary and sufficient conditions for the weighted strong law (1.11). In the general stationary case, however, proving weighted strong laws is a difficult problem and one needs restrictive conditions for such results. Let (w_k) be a sequence of positive reals and put $W_n = \sum_{k=1}^n w_k$. We say that $(w_k) \in \mathcal{W}$ if $W_n \to \infty$ and for every stationary ergodic sequence (Z_k) with finite means we have

$$W_n^{-1} \sum_{k=1}^n w_k Z_k \longrightarrow EZ_1 \qquad \text{a.s.} \tag{1.12}$$

Sufficient criteria for (1.12) will be given below. Using this terminology, we can formulate now our

Theorem 2. *Let $\{X_n, \mathcal{F}_n,\, n \in \mathbb{Z}\}$ be a stationary ergodic martingale difference sequence satisfying $EX_1^2 = 1$ and (1.8). Let (a_k) be a sequence of nonzero real numbers with $(a_k^2) \in \mathcal{W}$ and assume that*

$$B_n^2 := \sum_{k=1}^n a_k^2 \to \infty, \quad |a_n| = O(B_n^{1-\delta}) \quad a.s.$$

for some $\delta > 0$. Let $M(x) = \#\{k \geq 1 : |B_k/a_k| \leq x\}$ for $x > 0$ and assume that $M(x) \ll x^2$ a.s. as $x \to \infty$. Then the test (1.4)–(1.5) *holds with*

$$s_n^2 = \sum_{k=1}^n a_k^2 E[X_k^2|\mathcal{F}_{k-1}].$$

Note that Theorem 2 is new even in the case $a_k = 1$. By the ergodic theorem $(a_k^2) \in \mathcal{W}$ and since $B_n^2 = n$ we have $M(x) = x^2$. Hence we get

Corollary 1. *Let $\{X_n, \mathcal{F}_n, n \in \mathbb{Z}\}$ be a a stationary ergodic martingale difference sequence satisfying $EX_1^2 = 1$ and*

$$EX_1^2 \log\log |X_1| < \infty.$$

Then the test (1.4)–(1.5) holds with $s_n^2 = \sum_{k=1}^n E[X_k^2|\mathcal{F}_{k-1}]$.

In view of Feller's results in [8], the moment condition in Corollary 1 is best possible in the sense of the discussion following (1.8). Note that in Theorem 2 and Corollary 1 the scaling s_n in the upper-lower class test (1.4)–(1.5) is random, even though $s_n \sim B_n$ with a nonrandom B_n. This is a common feature of upper-lower class tests for martingales, see the remarks in Jain et al. [12], p. 127.

For general (a_k), a simple sufficient condition for the conclusion of Theorem 2 is

$$\frac{1}{B_n^2} \sum_{k=1}^{n-1} k|a_{k+1}^2 - a_k^2| = O(1), \tag{1.13}$$

as one can verify by simple calculations using the ergodic theorem. In particular (1.13) is satisfied if (a_n) is nonnegative, nondecreasing and regularly varying.

Theorem 2 will be deduced from a general integral test (Theorem 3) for martingale difference sequences which has some own interest.

Theorem 3. *Let $\{X_n, \mathcal{F}_n, n \in \mathbb{Z}\}$ be a martingale difference sequence with $EX_k^2 < \infty$ $(k \in \mathbb{Z})$. Assume that $B_n^2 := \sum_{k=1}^n EX_k^2 \to \infty$ and that for some $\delta > 0$ the following properties hold:*

(a) $\sum_{n=1}^\infty f(n)^{-4} E\left[X_n^4 I\{|X_n| \le \delta f(n)\}\right] < \infty$;

(b) $\sum_{n=1}^\infty f(n)^{-1} E\left[|X_n| I\{|X_n| > \delta f(n)\}\right] < \infty$;

(c) $B_n^{-2} \sum_{k=1}^n X_k^2 \to 1$ a.s.;

(d) $f^{-2}(n) \sum_{k=1}^n E[X_k^2 I\{|X_k| > \delta f(k)\}|\mathcal{F}_{k-1}] \to 0$,

where

$$f(k) = B_k / (\log\log B_k)^{1/2}.$$

Then the test (1.4)–(1.5) is valid with $s_n^2 = \sum_{k=1}^n E[X_k^2|\mathcal{F}_{k-1}]$.

The first to prove upper-lower class tests for martingales was Strassen [18] who proved that if $\{X_n, \mathcal{F}_n, n \in \mathbb{Z}\}$ is a martingale difference sequence with $EX_n^2 < \infty$ $(n \in \mathbb{Z})$, $S_n = X_1 + \ldots + X_n$ and $s_n^2 = \sum_{k=1}^n E[X_k^2|\mathcal{F}_{k-1}] \to \infty$ a.s. then the integral test (1.4)–(1.5) holds provided

$$|X_n| \le s_n/(\log s_n)^\alpha \qquad \alpha > 4. \tag{1.14}$$

Philipp and Stout [17] weakened (1.14) to

$$|X_n| \le s_n/(\log\log s_n)^{5/2}, \tag{1.15}$$

and Einmahl and Mason [5] showed that the test (1.4)–(1.5) actually holds under (1.3), a result which is optimal in view of Feller's theorem. The crucial new idea in [5] was to deduce the test (1.4)–(1.5) directly from the properties of the stopping times in the Skorohod representation for the partial sums $\sum_{k=1}^n X_k$, bypassing the strong approximation of the partial sum process, which had been used in all earlier results in the field.

Results for unbounded $\{X_n\}$ are far less satisfactory. Using truncation, from the result of Einmahl and Mason it follows that the test (1.4)–(1.5) holds provided

$$\sum_{k=1}^\infty s_k^{-2}(\log\log s_k)^3 E[X_k^2 I\{|X_k| > K s_k/(\log\log s_k)^{3/2}\}|\mathcal{F}_{k-1}] < \infty \text{ a.s.} \tag{1.16}$$

for some constant $K > 0$. Similar conditions (and sets of conditions) were given in Jain et al. [12] and Philipp and Stout [17]. While (1.16) is sharp in the case of bounded $\{X_n\}$ it is far from optimal in typical unbounded situations such as stationary $\{X_n\}$ where it requires $EX_n^2(\log X_n)^\gamma < \infty$ for some $\gamma > 0$. Our Theorem 3 requires in the stationary case only (1.8) which, as pointed out before, is optimal. The improvement is achieved by using the method of Einmahl and Mason and employing, in contrast to [18], [12], [17], a nonrandom truncation, an idea used first in the context of the LIL of Heyde and Scott [11].

2 Proofs

The following lemma is a consequence of the general theory of summation, see e.g. Hardy [10].

Lemma 1. *Let $d_n \ge 0$, $n = 1, 2, \ldots$ and assume that $D_n := \sum_{k=1}^n d_k \to \infty$, $d_n = o(D_n^{1-\delta})$ for some $\delta > 0$. If for some real sequence (x_n) the weighted averages $D_n^{-1} \sum_{k=1}^n d_k x_k$ converge to some $x \in \mathbb{R}$, then we also have*

$$\frac{\log\log D_n}{D_n} \sum_{k=1}^n \frac{d_k x_k}{\log\log D_k} \to x. \tag{2.1}$$

Proof of Theorem 1. We will deduce Theorem 1 from Theorem 3, whose proof will be given later. Assume (X_n) is an i.i.d. sequence satisfying $EX_1 = 0$, $EX_1^2 = 1$ and the moment condition (1.8) and assume that the arithmetic condition (1.9) holds. Applying Theorem 3 for the martingale difference sequence $a_n X_n$, the quantities B_n^2 and s_n^2 in Theorem 3 reduce to the quantity $s_n^2 = \sum_{k=1}^n a_k^2$ in Theorem 1 and thus it suffices to verify the following 4 conditions:

$$\sum_{n=1}^{\infty} \frac{a_n^4}{s_n^4} (\log\log s_n)^2 E[X_1^4 I\{|X_1| \le |s_n/a_n| (\log\log s_n)^{-1/2}\}] < \infty \tag{2.2}$$

$$\sum_{n=1}^{\infty} \frac{|a_n|}{s_n} (\log\log s_n)^{1/2} E[|X_1| I\{|X_1| \ge |s_n/a_n| (\log\log s_n)^{-1/2}\}] < \infty \tag{2.3}$$

$$s_n^{-2} \sum_{k=1}^{n} a_k^2 X_k^2 \to 1 \quad \text{a.s.} \tag{2.4}$$

$$\frac{\log\log s_n}{s_n^2} \sum_{k=1}^{n} a_k^2 E[X_1^2 I\{|X_1| \ge |s_k/a_k| (\log\log s_k)^{-1/2}\}] \to 0. \tag{2.5}$$

The validity of (2.4) is an immediate consequence of (1.9) and the strong law of large numbers of Jamison et al. [13]. To prove (2.2) and (2.3) introduce

$$N(x) = \#\{k \ge 1 : |s_k/a_k| (\log\log s_k)^{-1/2} \le x\}.$$

Let us note that $|a_n| = O(s_n^{1-\delta})$ implies

$$\log\log s_n \ll \log\log |s_n/a_n| \tag{2.6}$$

and consequently each of the inequalities

$$s_n^2/(a_n^2 \log\log s_n) \le x^2, \qquad \frac{s_n^2/a_n^2}{\log\log(s_n^2/a_n^2)} \ll x^2, \qquad s_n^2/a_n^2 \ll x^2 \log\log x$$

implies the next one. Thus (1.9) implies that

$$N(x) \ll x^2 \log\log x \qquad \text{as } x \to \infty. \tag{2.7}$$

Clearly, the sum in (2.2) equals $\int_{-\infty}^{\infty} x^4 A(x) dF(x)$ where F is the common distribution function of the X_n's and

$$A(x) = \sum_{n=1}^{\infty} \frac{a_n^4}{s_n^4} (\log\log s_n)^2 I(|x| \le |s_n/a_n| (\log\log s_n)^{-1/2}) = \int_{|x|}^{\infty} \frac{dN(t)}{t^4}.$$

$A(x)$ is an even function and integration by parts gives for $x > 0$

$$\int_x^\infty \frac{dN(t)}{t^4} = -\frac{N(x)}{x^4} + 4\int_x^\infty \frac{N(t)}{t^5}dt. \tag{2.8}$$

(Since the total mass of N over $(0,\infty)$ is infinite, to make the proof of (2.8) precise, one has to apply integration by parts over (x,T) and then let $T \to \infty$, using the fact that $\int_x^\infty N(t)/t^5 dt < \infty$ and $N(T)/T^4 \to 0$ by (2.7).) By (2.7) the right hand side of (2.8) is $O(x^{-2}\log\log x)$ and thus the sum in (2.2) is

$$\int_{-\infty}^\infty x^4 A(x)dF(x) \ll \int_{-\infty}^\infty x^2 \log\log|x|\, dF(x) < \infty$$

by (1.8). Next we prove (2.3). Let us note that by $|a_n| = O(s_n^{1-\delta})$ we have

$$|s_n/a_n|(\log\log s_n)^{-1/2} \geq s_n^{\delta/2} \tag{2.9}$$

for sufficiently large n and thus we can change finitely many of the a_n's so that $|s_n/a_n|(\log\log s_n)^{-1/2} > 1$ for $n \geq 1$. Consequently, we can assume that $N(x) = 0$ for $0 \leq x \leq 1$. Similarly as before, the sum in (2.3) is $\int_{-\infty}^\infty |x|B(x)dF(x)$ where

$$\begin{aligned} B(x) &= \sum_{n=1}^\infty \frac{|a_n|}{s_n}(\log\log s_n)^{1/2} I(|x| \geq |s_n/a_n|(\log\log s_n)^{-1/2}) \\ &= \int_1^x \frac{dN(t)}{t} = \frac{N(x)}{x} + \int_1^x \frac{N(t)}{t^2}dt = O(x\log\log x). \end{aligned}$$

Hence the sum in (2.3) is $\int_{-\infty}^\infty |x|B(x)dF(x) \ll \int_{-\infty}^\infty x^2\log\log|x|\,dF(x) < \infty$ by (1.8). It remains to show (2.5). By (2.6) and (2.9) we get

$$\begin{aligned} &\frac{\log\log s_n}{s_n^2}\sum_{k=1}^n a_k^2 E[X_1^2 I\{|X_1| \geq |s_k/a_k|(\log\log s_k)^{-1/2}\}] \\ &\ll \frac{\log\log s_n}{s_n^2}\sum_{k=1}^n \frac{a_k^2}{\log\log s_k} E[X_1^2 \log\log X_1^2 I\{|X_1| \geq |s_k/a_k|(\log\log s_k)^{-1/2}\}]. \end{aligned}$$

Using (1.8) and (2.9) it follows that

$$E[X_1^2\log\log X_1^2 I\{|X_1| \geq |s_k/a_k|(\log\log s_k)^{-1/2}\}] \to 0$$

and thus (2.5) is an immediate consequence of Lemma 1.

To prove the converse part of Theorem 1, let (a_n) be a sequence of real numbers satisfying the assumptions of the theorem and assume that for any i.i.d. sequence (X_n) with

$$EX_1 = 0, \quad EX_1^2 = 1, \quad EX_1^2\log\log|X_1| < \infty$$

the partial sums $\sum_{k=1}^n a_kX_k$ satisfy the test (1.4)–(1.5), but (1.10) fails. Let

$$N^*(x) = \#\{k \ge 1 : |s_k/a_k|(\log\log s_k)^{1/2} \le x\}.$$

Relation (2.6) shows that the inequality $|s_k/a_k| \le x$ implies

$$|s_k/a_k|(\log\log s_k)^{1/2} \le Cx(\log\log x)^{1/2}$$

for some constant C and consequently

$$M(x) \le N^*(Cx(\log\log x)^{1/2}).$$

Thus if (1.10) fails, then

$$N^*(x) \ll x^2 \log\log x$$

cannot be valid, either. Thus there exists an increasing sequence (x_k) of positive numbers with $x_1 = 1/100$ and $x_k \to \infty$ such that

$$N^*(x_k)/x_k^2 \log\log x_k \to \infty.$$

Then there exists a sequence (f_k) of positive numbers such that

$$\sum_{k=1}^{\infty} f_k x_k^2 \log\log x_k < \infty, \qquad \sum_{k=1}^{\infty} f_k N^*(x_k) = \infty.$$

In particular $\sum_{k=1}^\infty f_k < \infty$ and thus by scaling (f_k) we can assume that $\sum_{k=1}^\infty f_k = 1$. For a fixed $r \ge 1$ define (f_n^*) so that $f_1^* = f_1 + \ldots + f_r$, $f_2^* = \ldots = f_r^* = 0$ and $f_n^* = f_n$ for $n > r$. Clearly $\sum_{n=1}^\infty f_n^* x_n^2 = (f_1 + \ldots + f_r)x_1^2 + \sum_{n>r} f_n x_n^2 \le x_1^2 + \sum_{n>r} f_n x_n^2 \log\log x_n < 2 \cdot 10^{-4}$ provided we choose r so large that $\sum_{n>r} f_n x_n^2 \log\log x_n \le 10^{-4}$. Hence without loss of generality we may assume that $\sum_{n=1}^\infty f_n x_n^2 \le 2 \cdot 10^{-4}$. Let (Y_n) be i.i.d. random variables with $P(Y_1 = x_k) = P(Y_1 = -x_k) = \frac{1}{2} f_k$ $(k = 1, 2, \ldots)$; clearly $EY_1 = 0$, $EY_1^2 \le 2 \cdot 10^{-4}$, $EY_1^2 \log\log |Y_1| < \infty$, $EN^*(|Y_1|) = \infty$. Let $X_n = cY_n$ where $c \ge 10$ is chosen so that $EX_1^2 = 1$; clearly $EX_1 = 0$, $EX_1^2 \log\log |X_1| < \infty$. Let F denote the distribution function of Y_1. We claim that a_nX_n cannot satisfy the LIL

$$\limsup_{n\to\infty} (2s_n^2 \log\log s_n)^{1/2} \sum_{k=1}^n a_kX_k = 1 \quad \text{a.s.} \tag{2.10}$$

and thus the test (1.4)–(1.5) for a_nX_n also fails. Indeed, if (2.10) were true, then using $s_{n+1}/s_n \to 1$ (which follows from $a_n/s_n \to 0$) we would have almost surely for sufficiently large n

$$\left|\sum_{k=1}^n a_kX_k\right| < 4(s_n^2 \log\log s_n)^{1/2}, \qquad \left|\sum_{k=1}^{n-1} a_kX_k\right| < 4(s_n^2 \log\log s_n)^{1/2}$$

and consequently

$$P(|a_nX_n| \geq 8(s_n^2 \log\log s_n)^{1/2} \text{ i.o.}) = 0.$$

Thus by the Borel-Cantelli lemma and $c \geq 10$

$$\sum_{n=1}^{\infty} P(|a_nX_n| \geq c(s_n^2 \log\log s_n)^{1/2}) < \infty. \tag{2.11}$$

But the last sum equals

$$\begin{aligned}
&\sum_{n=1}^{\infty} EI\{|Y_n| \geq |s_n/a_n|(\log\log s_n)^{1/2}\} \\
&= \int_{-\infty}^{\infty} \left[\sum_{n=1}^{\infty} I(|x| \geq |s_n/a_n|(\log\log s_n)^{1/2})\right] dF(x) \\
&= \int_{-\infty}^{\infty} N^*(|x|)dF(x) = EN^*(|Y_1|) = \infty,
\end{aligned}$$

contradicting to (2.11).

Proof of Theorem 2. We will deduce Theorem 2 also from Theorem 3. The verification of conditions (**a**), (**b**) of Theorem 3 (which leads again to (2.2), (2.3)) remains unchanged. The validity of (**c**) follows from $(a_k^2) \in \mathcal{W}$. Finally, verifying condition (**d**) for $\delta = 1$ requires to show that

$$\frac{\log\log B_n}{B_n^2} \sum_{k=1}^{n} a_k^2 E[X_k^2 I\{|X_k| \geq |B_k/a_k|(\log\log B_k)^{-1/2}\}|\mathcal{F}_{k-1}] \to 0 \quad \text{a.s.} \tag{2.12}$$

Let

$$Z_k(m) := E[X_k^2 \log\log X_k^2 I\{|X_k| \geq |B_m/a_m|(\log\log B_m)^{-1/2}\}|\mathcal{F}_{k-1}].$$

By the assumption $|a_n| = O(B_n^{1-\delta})$, the inequality $|X_k| \geq |B_k/a_k|(\log\log B_k)^{-1/2}$ implies $|X_k| \geq cB_k^{\delta/2}$ and consequently $\log\log X_k^2 \geq c \log\log B_k$ with some constant $c > 0$. Thus the left hand side of (2.12) is bounded by

$$\text{const} \cdot \frac{\log\log B_n}{B_n^2} \sum_{k=1}^{n} \frac{a_k^2}{\log\log B_k} Z_k(k),$$

with a constant independent of n. Observe that the sequence $\{Z_k(m), k \geq 1\}$ is stationary and ergodic for every fixed m and thus by $(a_k^2) \in \mathcal{W}$ and Lemma 1 we have almost surely for every $\varepsilon > 0$

$$\frac{\log\log B_n}{B_n^2} \sum_{k=1}^{n} \frac{a_k^2}{\log\log B_k} Z_k(m) \longrightarrow EZ_1(m) < \varepsilon,$$

if m is chosen large enough. Clearly $\{Z_k(m), m \geq 1\}$ is non-increasing for every k and hence (2.12) follows immediately.

The fact that the sequence $a_n = f(n)$ in Example 1 satisfies condition (1.9) is implicit in Berkes and Weber [2], where it is proved that under the same conditions on f, the weighted i.i.d. sums $\sum_{k\leq N} f(k)X_k$ satisfy the LIL under $EX_1^2 = 0$, $EX_1^2 = 1$. Proving (1.9), i.e.

$$M(t) := \#\{n \geq 1 : \sum_{k\leq n} f^2(k) \leq t f^2(n)\} \ll t^2 \quad \text{as } t \to \infty, \tag{2.13}$$

directly in this case seems to be difficult and in Berkes and Weber [2] an indirect argument was used, showing that (2.13) is equivalent to the validity of

$$E\, X^4 \int_{y\geq X^2} \frac{M(y)}{y^3} dy < \infty$$

for all r.v.'s X with $EX^2 < \infty$. The last relation can be verified by a randomization argument (see [2], pp. 1229-1231) using elementary arithmetic properties of additive functions.

To verify the statement in Example 2, let (ξ_n) be a stationary ergodic sequence with $0 < E|\xi_1|^p < \infty$ for some $p > 2$. Then one can find a probability space $(\Omega, \mathcal{A}, P)$, a function $f \in L^p(\Omega, \mathcal{A}, P)$ and an ergodic transformation $\tau : \Omega \to \Omega$ such that the sequence (η_n) of random variables defined by $\eta_n(\omega) = f(\tau^n \omega)$ has the same distribution as (ξ_n). By a result of Assani [1] we have, letting $g = f^2$,

$$\lim_{t\to\infty} \frac{\#\big\{n : g(\tau^n\omega)/n \geq 1/t\big\}}{t} = \int_\Omega g dP \qquad \text{a.s.}$$

Using the ergodic theorem, the last relation implies, letting $D = \int_\Omega f^2 dP > 0$,

$$\begin{aligned} &\lim_{t\to\infty} \frac{\#\big\{n : \big(\sum_{k=1}^n f^2(\tau^k\omega)\big)^{1/2} / f(\tau^n\omega) \leq t\big\}}{t^2} \\ &= \lim_{t\to\infty} \frac{\#\big\{n : Dn(1+o(1))/f^2(\tau^n\omega) \leq t^2\big\}}{t^2} = 1 \qquad \text{a.s.} \end{aligned}$$

which shows that the sequence (η_n), and thus also (ξ_n), satisfies (1.9). On the other hand, $E|\xi_1|^p < \infty$, $p > 2$ and the Borel-Cantelli lemma imply that $|\xi_n| = O(n^{(1-\delta)/2})$ a.s. with a suitable $\delta > 0$ and thus using the ergodic theorem and $E\xi_1^2 > 0$ it follows that

$$|\xi_n| = O\left(\sum_{k-1}^n \xi_n^2\right)^{(1-\delta)/2} \qquad \text{a.s.}$$

Hence (ξ_n) satisfies the coefficient condition $|a_n| = O(s_n^{1-\delta})$ of Theorem 1.

Proof of Theorem 3. We start with defining a truncated MDS $\{X_k^*, \mathcal{F}_k^*, k \ge 1\}$ as follows:

$$X_k^* = X_k I\{|X_k| \le \delta f(k)\} - E[X_k I\{|X_k| \le \delta f(k)\}|\mathcal{F}_{k-1}^*], \tag{2.14}$$

where

$$f(k) = B_k/(\log\log B_k)^{1/2}$$

is the function defined in Theorem 3. Here $\mathcal{F}_k^* = \sigma(X_1^*, \ldots, X_k^*)$ if $k \ge 1$ and $\mathcal{F}_0^* = \{\emptyset, \Omega\}$. (Note that starting with $\mathcal{F}_0^*$, relation (2.14) determines successively X_1^*, $\mathcal{F}_1^*$, X_2^*, $\mathcal{F}_2^*, \ldots$.) Denote by S_n^* the partial sum $X_1^* + \cdots + X_n^*$ and similarly to s_n^2, set

$${s_n^*}^2 = \sum_{k=1}^n E[{X_k^*}^2|\mathcal{F}_{k-1}^*].$$

Finally set $X_k^{**} = X_k - X_k^*$ and $S_n^{**} = X_1^{**} + \cdots + X_n^{**}$. Clearly

$$|E[X_k I\{|X_k| \le \delta f(k)\}|\mathcal{F}_{k-1}^*]| = |E[X_k I\{|X_k| > \delta f(k)\}|\mathcal{F}_{k-1}^*]| \quad \text{a.s.} \tag{2.15}$$

Lemma 2. *Under condition* (**b**) *we have*

$$f(n)^{-2} \sum_{k=1}^n X_k^2 I\{|X_k| > \delta f(k)\} \to 0 \quad \text{a.s.}$$

Proof. We have

$$\begin{aligned} \sum_{k=1}^n X_k^2 I\{|X_k| > \delta f(k)\} &= \sum_{k=1}^n X_k^2 I^2\{|X_k| > \delta f(k)\} \\ &\le \left(\sum_{k=1}^n |X_k| I\{|X_k| > \delta f(k)\} \right)^2, \end{aligned}$$

and thus the result follows from (**b**) and Kronecker's lemma. ☐

In the sequel, $c_n \sim d_n$ means $\lim_{n\to\infty} c_n/d_n = 1$.

Lemma 3. *Under conditions* (**a**)–(**d**) *we have*

$${s_n^*}^2 \sim B_n^2 \sim s_n^2 \quad \text{a.s.}$$

Proof. We will show that ${s_n^*}^2 \sim B_n^2$; the proof of the second relation is similar. An easy calculation gives

$$B_n^{-2} \sum_{k=1}^{n} E[X_k^{*2} | \mathcal{F}_{k-1}^*] \tag{2.16}$$
$$= B_n^{-2} \sum_{k=1}^{n} E[X_k^2 I\{|X_k| \le \delta f(k)\} | \mathcal{F}_{k-1}^*] - B_n^{-2} \sum_{k=1}^{n} (E[X_k I\{|X_k| \le \delta f(k)\} | \mathcal{F}_{k-1}^*])^2.$$

By (2.15) we obtain

$$B_k^{-2}(E[X_k I\{|X_k| \le \delta f(k)\} | \mathcal{F}_{k-1}^*])^2$$
$$\le \delta (f(k) \log\log B_k)^{-1} E[|X_k| I\{|X_k| > \delta f(k)\} | \mathcal{F}_{k-1}^*],$$

and thus by Kronecker's lemma and (**b**) the second term in (2.16) tends to zero. Set

$$Y_k := X_k^2 I\{|X_k| \le \delta f(k)\} - E[X_k^2 I\{|X_k| \le \delta f(k)\} | \mathcal{F}_{k-1}^*].$$

Clearly (Y_k) is a MDS and since by (**a**)

$$\sum_{k=1}^{\infty} B_k^{-4} E Y_k^2 \le \sum_{k=1}^{\infty} B_k^{-4} E[X_k^4 I\{|X_k| \le \delta f(k)\}] < \infty,$$

the martingale convergence theorem and Kronecker's lemma imply that $B_n^{-2} \sum_{k=1}^n Y_k \to 0$ almost surely, i.e.

$$B_n^{-2} \sum_{k=1}^{n} \left(X_k^2 I\{|X_k| \le \delta f(k)\} - E[X_k^2 I\{|X_k| \le \delta f(k)\} | \mathcal{F}_{k-1}^*] \right) \to 0 \quad \text{a.s.}$$

In view of (**c**), (2.16) and the last relation, it remains to show that

$$B_n^{-2} \sum_{k=1}^{n} X_k^2 I\{|X_k| > \delta f(k)\} \to 0 \quad \text{a.s.},$$

which follows from Lemma 2. □

Lemma 4. *Under conditions* (**a**)–(**b**) *we have*

$$\sum_{k=1}^{\infty} f(k)^{-4} E X_k^{*4} < \infty \quad \text{a.s.}$$

Proof. By (2.15) we have

$$\sum_{k=1}^{\infty} f(k)^{-4} E X_k^{*4}$$
$$= \sum_{k=1}^{\infty} f(k)^{-4} E(X_k I\{|X_k| \le \delta f(k)\} - E[X_k I\{|X_k| \le \delta f(k)\}|\mathcal{F}^*_{k-1}])^4$$
$$\le \sum_{k=1}^{\infty} f(k)^{-4} \left(E[X_k^4 I\{|X_k| \le \delta f(k)\}] + 15\delta^3 f(k)^3 E|E[X_k I\{|X_k| \le \delta f(k)\}|\mathcal{F}^*_{k-1}]|\right)$$
$$\le \sum_{k=1}^{\infty} f(k)^{-4} \left(E[X_k^4 I\{|X_k| \le \delta f(k)\}] + 15\delta^3 f(k)^3 E[|X_k| I\{|X_k| > \delta f(k)\}]\right),$$

and the last sum is finite by (**a**) and (**b**). □

Lemma 5. *Under conditions* (**a**)–(**d**) *we have*

$$\frac{\log\log s_n}{s_n^2}(s_n^2 - s_n^{*\,2}) \to 0 \quad \text{a.s.}$$

Proof. By Lemma 3 it suffices to show that

$$f(n)^{-2}(s_n^2 - s_n^{*\,2}) \to 0 \quad \text{a.s.}$$

We can write

$$s_n^2 - s_n^{*\,2} = \sum_{k=1}^{n} \left(E[X_k^2|\mathcal{F}_{k-1}] - E[X_k^{*2}|\mathcal{F}_{k-1}]\right)$$
$$+ \sum_{k=1}^{n} \left(E[X_k^{*2}|\mathcal{F}_{k-1}] - E[X_k^{*2}|\mathcal{F}^*_{k-1}]\right) := S_n^{(1)} + S_n^{(2)} \quad \text{(say)}.$$

A straightforward calculation shows (using again (2.15)) that

$$|E[X_k^2 - X_k^{*2}|\mathcal{F}_{k-1}]|$$
$$\le E[X_k^2 I\{|X_k| > \delta f(k)\}|\mathcal{F}_{k-1}] + (E[X_k I\{|X_k| \le \delta f(k)\}|\mathcal{F}^*_{k-1}])^2$$
$$+2|E[X_k I\{|X_k| \le \delta f(k)\}|\mathcal{F}_{k-1}]||E[X_k I\{|X_k| \le \delta f(k)\}|\mathcal{F}^*_{k-1}]|$$
$$\le E[X_k^2 I\{|X_k| > \delta f(k)\}|\mathcal{F}_{k-1}] + 3\delta f(k) E[|X_k| I\{|X_k| > \delta f(k)\}|\mathcal{F}^*_{k-1}].$$

Thus conditions (**d**) and (**b**), in connection with Kronecker's lemma and the Beppo Levi theorem imply that

$$f(n)^{-2} S_n^{(1)} \to 0 \quad \text{a.s.}$$

It remains to prove

$$f(n)^{-2}S_n^{(2)} \to 0 \quad \text{a.s.},$$

which follows from Lemma 4 by a similar argument, observing that $E(X_k^{*2}|\mathcal{F}_{k-1}) - E(X_k^{*2}|\mathcal{F}_{k-1}^*)$ is a martingale difference sequence and applying Kronecker's lemma and the martingale convergence theorem. □

By a standard argument in upper-lower class theory (see Feller [8], Lemma 1) we can assume that the function φ in the test (1.4)–(1.5) satisfies

$$a\sqrt{\log\log t} \le \varphi(t) \le b\sqrt{\log\log t} \tag{2.17}$$

for some positive constants $a < b$ and in particular φ tends to ∞. Recall also that φ is a nondecreasing function. Clearly we can write

$$\varphi(t) = \tilde{\varphi}(t) + \frac{1}{\tilde{\varphi}(t)} \quad \text{with} \quad \tilde{\varphi}(t) = \frac{\varphi(t)}{2} + \left(\frac{\varphi^2(t)}{4} - 1\right)^{1/2}, \tag{2.18}$$

provided $\varphi(t) \ge 2$. By another standard observation in the theory, $I(\varphi)$ is finite if and only if $I(\tilde{\varphi})$ is finite. The next lemma shows it is enough to consider only functions φ which are smooth in some sense.

Lemma 6. *Assume that $I(\varphi) = \infty (< \infty)$. Then there is some nondecreasing function $\hat{\varphi} \ge \varphi (\le \varphi)$ and some absolute constant A such that $I(\hat{\varphi}) = \infty (< \infty)$ and*

$$|\hat{\varphi}(x) - \hat{\varphi}(y)| \le A \cdot \frac{\hat{\varphi}(x)}{x}|x-y| \quad \text{if } x < y \text{ and } [y, 2y] \cap [x, 2x] \ne \emptyset. \tag{2.19}$$

Proof. Assume that $I(\varphi) = \infty$. Some simple calculations show that

$$\hat{\varphi}(x) = \frac{1}{x}\int_x^{2x} \varphi(t)\, dt \quad (x > x_0).$$

will work. The case $I(\varphi) < \infty$ can be treated similarly by defining $\hat{\varphi}(x) = \int_{x/2}^{x} \varphi(t)\, dt$. □

Next we observe that

$$|X_n^*| \le K_n s_n^*, \tag{2.20}$$

where $K_n = 2\delta f(n)/s_n^* \sim 2\delta(\log\log s_n^*)^{-1/2}$ by Lemma 3; also K_n is $\mathcal{F}_{n-1}^*$ measurable. By the martingale version of the Skorokhod embedding theorem (Strassen [18]) we can assume that the sequence (X_n^*) is defined on a probability space together with a standard Wiener process $\{W(t),\, t \geq 0\}$ such that

$$S_n^* = W(T_n), \quad \text{where} \quad T_n = \sum_{m=1}^{n} \tau_m, \tag{2.21}$$

where τ_n are non-negative r.v.'s, τ_n is $\mathcal{F}_n^*$ measurable $(n = 1, 2, \ldots)$ and

$$E[\tau_n | \mathcal{F}_{n-1}^*] = E[X_n^{*2} | \mathcal{F}_{n-1}^*] \quad \text{a.s.} \tag{2.22}$$

Also we have for any $r \geq 1$

$$E[\tau_n^r | \mathcal{F}_{n-1}^*] \leq L_r E[X_n^{*2r} | \mathcal{F}_{n-1}^*] \quad \text{a.s.}, \tag{2.23}$$

where L_r is a constant depending only on r and moreover, for $T_n \leq t \leq T_{n+1}$

$$|W(t) - W(T_n)| \leq K_{n+1} s_{n+1}^*. \tag{2.24}$$

The last relation is a consequence of the construction of the Skorokhod stopping times and plays a crucial role in the following lemma, implicit in the proof of Theorem 1.1 of Einmahl and Mason [5].

Lemma 7. *Assume that $\{X_n^*, \mathcal{F}_n^*,\, n \geq 1\}$ is a MDS with finite variances such that $s_n^{*\,2} := \sum_{k=1}^n E[X_k^{*2} | \mathcal{F}_{k-1}^*] \to \infty$. Assume that (2.20)–(2.24) hold with some $K_n \sim \mathrm{const} \cdot (\log \log s_n^*)^{-1/2}$ and K_n is $\mathcal{F}_{n-1}^*$ measurable. If there exists some positive constant K such that*

$$\limsup_{n \to \infty} \frac{\log \log s_n^{*2}}{s_n^{*\,2}} |T_n - s_n^{*\,2}| \leq K \quad \text{a.s.}, \tag{2.25}$$

then for every positive and nondecreasing function φ

$$P(S_k^* > s_k^* \varphi(s_k^{*2}) \text{ i.o.}) = \begin{cases} 1 & \text{if } I(\varphi) = \infty, \\ 0 & \text{if } I(\varphi) < \infty. \end{cases}$$

Note that the assumptions of Lemma 7 imply that $s_n^* \sim s_{n+1}^*$. Thus it follows from (2.24) and (2.25) that for $T_n \leq t \leq T_{n+1}$ and for sufficiently large n

$$|W(t) - W(T_n)| \leq \mathrm{const} \cdot \sqrt{t} / (\log \log t)^{1/2} \quad \text{a.s.} \tag{2.26}$$

Now the proof of Theorem 1.1 of Einmahl and Mason [5] can be followed almost verbatim, observing that the argument still goes through if their assumption (1.3) is replaced by (2.25).

The remainder of the proof of Theorem 3 will be divided into two steps. In the first step we will show that the integral test (1.4)–(1.5) holds for the truncated MDS $\{X_k^*, \mathcal{F}_k^*, k \geq 1\}$. Then we will show that if φ is monotone and satisfies the smoothness condition (2.19), then

$$P(S_k > s_k \varphi(s_k^2) \text{ i.o.}) = P(S_k^* > s_k^* \varphi({s_k^*}^2) \text{ i.o.}). \tag{2.27}$$

Now if $I(\varphi) = \infty$, the $\hat{\varphi}$ in Lemma 6 satisfies $\hat{\varphi} \geq \varphi$, $I(\hat{\varphi}) = \infty$ and thus

$$1 = P(S_k^* > s_k^* \hat{\varphi}(s_k^{*2}) \text{ i.o.}) = P(S_k > s_k \hat{\varphi}(s_k^2) \text{ i.o.}) \leq P(S_k > s_k \varphi(s_k^2) \text{ i.o.}).$$

An analogous result holds if $I(\varphi) < \infty$.

Step 1. By Lemma 3 and Lemma 7 it suffices to show that

$$|T_n - {s_n^*}^2| = o\left(f(n)^2\right) \quad \text{a.s.} \tag{2.28}$$

Using the definition of ${s_n^*}^2$ and (2.21), (2.22) we have

$$T_n - {s_n^*}^2 = \sum_{k=1}^n (\tau_k - E[X_k^{*2} | \mathcal{F}_{k-1}^*]) = \sum_{k=1}^n (\tau_k - E[\tau_k | \mathcal{F}_{k-1}^*]) \quad \text{a.s.}$$

By (2.23) and Lemma 4 we have

$$\begin{aligned} &\sum_{k=1}^\infty f(k)^{-4} E(\tau_k - E[\tau_k | \mathcal{F}_{k-1}^*])^2 \\ &\qquad \leq \sum_{k=1}^\infty f(k)^{-4} E\tau_k^2 \leq L_2 \sum_{k=1}^\infty f(k)^{-4} E X_k^{*4} < \infty, \end{aligned}$$

and thus by the martingale convergence theorem the series

$$\sum_{k=1}^\infty f(k)^{-2} (\tau_k - E[\tau_k | \mathcal{F}_{k-1}^*])$$

is a.s. convergent, implying (2.28) by the Kronecker lemma.

Step 2. Define $\tilde{\varphi}$ as in (2.18) and set

$$R_k = \frac{s_k}{\tilde{\varphi}(s_k^2)} + (\tilde{\varphi}(s_k^2) s_k - \tilde{\varphi}({s_k^*}^2) s_k^*).$$

Then we have, on one hand,

$$\begin{aligned} &P(S_k > s_k \varphi(s_k^2) \text{ i.o.}) \\ &\qquad = P(S_k^* + S_k^{**} > s_k^* \tilde{\varphi}({s_k^*}^2) + R_k \text{ i.o.}) \\ &\qquad \leq P(S_k^* > s_k^* \tilde{\varphi}({s_k^*}^2) \text{ i.o.}) + P(|S_k^{**}| > R_k \text{ i.o.}). \end{aligned}$$

and on the other hand,

$$\begin{aligned}
&P(S_k > s_k\varphi(s_k^2) \text{ i.o.})\\
&\quad \ge P(S_k^* > s_k^*\tilde\varphi(s_k^{*2}) + |R_k - S_k^{**}| \text{ i.o.})\\
&\quad = P(S_k^* > s_k^*\tilde{\tilde\varphi}(s_k^{*2}) + s_k^*\tilde{\tilde\varphi}(s_k^{*2})^{-1} + |R_k - S_k^{**}| \text{ i.o.}).
\end{aligned}$$

We now show that for any $\varepsilon > 0$

$$P(|S_k^{**}| > \varepsilon R_k \text{ i.o.}) = 0 \quad \text{and} \quad |R_k - S_k^{**}| \le \frac{(\kappa-1)}{2} s_k^*\tilde{\tilde\varphi}(s_k^{*2})^{-1} \quad \text{a.s.} \tag{2.29}$$

for some large enough κ, which implies in view of the forgoing estimates that

$$\begin{aligned}
&P(S_k^* > \tilde{\tilde\varphi}(s_k^{*2})s_k^* + \kappa s_k^*\tilde{\tilde\varphi}(s_k^{*2})^{-1} \text{ i.o.})\\
&\quad \le P(S_k > \varphi(s_k^2)s_k \text{ i.o.}) \le P(S_k^* > \tilde\varphi(s_k^{*2})s_k^* \text{ i.o.}).
\end{aligned}$$

(Here $\tilde{\tilde\varphi}$ is obtained by iterating the operation in (2.18).) Since $I(\varphi) = \infty$ if and only if $I(\tilde\varphi) = \infty$ and $I(\tilde{\tilde\varphi} + \kappa/\tilde{\tilde\varphi}) = \infty$, we get (2.27). To prove (2.29) we start with showing that the dominating part in R_k is $s_k/\tilde\varphi(s_k^2)$, i.e.

$$\frac{\tilde\varphi(s_k^2)}{s_k}(\tilde\varphi(s_k^2)s_k - \tilde\varphi(s_k^{*2})s_k^*) \to 0 \quad \text{a.s.} \tag{2.30}$$

In view of Lemma 3 and

$$\tilde\varphi(s_k^2)s_k - \tilde\varphi(s_k^{*2})s_k^* = \frac{\tilde\varphi(s_k^2)(s_k^2 - s_k^{*2})}{s_k + s_k^*} + s_k^*(\tilde\varphi(s_k^2) - \tilde\varphi(s_k^{*2}))$$

this will follow if

$$\frac{\tilde\varphi(s_k^2)^2}{s_k^2}(s_k^2 - s_k^{*2}) \to 0 \quad \text{a.s.} \quad \text{and} \quad \tilde\varphi(s_k^2)(\tilde\varphi(s_k^2) - \tilde\varphi(s_k^{*2})) \to 0 \quad \text{a.s.} \tag{2.31}$$

Now the first relation in (2.31) follows from Lemma 5 and (2.17), since (2.17) remains valid for $\tilde\varphi$. Note that $\varphi(t) \sim \tilde\varphi(t)$ by (2.18). As we noted earlier, we can assume that $\tilde\varphi$ satisfies (2.19). Clearly, since $s_k^{*2} \sim s_k^2$ the intervals $[s_k^2, 2s_k^2]$ and $[s_k^{*2}, 2s_k^{*2}]$ will not be disjoint for any $k \ge k_0(\omega)$, where k_0 is almost surely finite. Thus we get from (2.19)

$$\tilde\varphi(s_k^2)(\tilde\varphi(s_k^2) - \tilde\varphi(s_k^{*2})) \le A\frac{\tilde\varphi(s_k^2)^2}{s_k^2}(s_k^2 - s_k^{*2}) \quad \text{for all } k \ge k_0,$$

with an absolute constant A and here the right hand side tends to zero as we have already noted. Thus we proved that the second relation of (2.31) is also valid and thus the dominating part of R_k is $s_k/\tilde\varphi(s_k^2)$. Hence to prove the first relation in (2.29) it suffices to show

$$|S_k^{**}| = o\left(\frac{s_k}{\tilde\varphi(s_k^2)}\right) \quad \text{a.s.} \tag{2.32}$$

which by Lemma 3 and (2.17) will follow if

$$|S_k^{**}| = o\,(f(k)) \quad \text{a.s.} \tag{2.33}$$

It easy to see (cf. (2.15)) that $E|X_k^{**}| \le 2\,E\,[|X_k| I\{|X_k| > \delta f(k)\}]$ and hence (2.33) follows from condition (**b**) and Kronecker's lemma.

Since we proved that $|S_k^{**}| = o(R_k)$ a.s. and that the dominating part of R_k is $s_k/\tilde{\varphi}(s_k^2)$, we have in view of (2.17), Lemma 3 and $\varphi(t) \sim \tilde{\varphi}(t)$ $(t \to \infty)$,

$$\begin{aligned} |R_k - S_k^{**}| &= |R_k|(1 + o(1)) \\ &= \frac{s_k}{\tilde{\varphi}(s_k^2)}(1 + o(1)) \le \text{const} \cdot \frac{s_k^*}{\tilde{\tilde{\varphi}}(s_k^{*2})} \quad \text{a.s.} \end{aligned}$$

This proves the second relation of (2.29).

References

[1] I. Assani, Convergence of the p-series for stationary sequences. New York J. of Math. **3A** (1997/98), 15-30.

[2] I. Berkes and M. Weber, A law of the iterated logarithm for arithmetic functions. Proc. Amer. Math. Soc. **135** (2007), 126–135.

[3] U. Einmahl, The Darling-Erdős theorem for sums of i.i.d. random variables. Probab. Theory Related Fields **82** (1989), 241–257.

[4] U. Einmahl and D.M. Mason, Darling-Erdős theorems for martingales. J. Theoret. Probab. **2** (1989), 437–460.

[5] U. Einmahl and D.M. Mason, Some results on the almost sure behavior of martingales. Limit theorems in probability and statistics (Pécs 1989), pp. 185–195, Colloquia Math. Soc. János Bolyai, 57, North-Holland, Amsterdam, 1990.

[6] P. Erdős and M. Kac, The Gaussian law of errors in the theory of additive number-theoretic functions. Amer. J. Math. **62** (1940), 738-742.

[7] W. Feller, The general form of the so-called law of the iterated logarithm. Trans. Amer. Math. Soc. **54** (1943), 373–402.

[8] W. Feller, The law of the iterated logarithm for identically distributed random variables. Ann. of Math. **47** (1946), 631–638.

[9] E. Fisher, A Skorohod representation and an invariance principle for sums of weighted i.i.d. random variables, Rocky Mount. J. Math. **22** (1992), 169-179.

[10] G. H. Hardy, The second theorem of consistency for summable series, Proc. London Math. Soc. **15** (1916), 72–88.

[11] C. C. Heyde and D. J. Scott, Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments. Ann. Probab. **1** (1973), 428–436.

[12] N. C. Jain, K. Jogdeo and W. Stout, Upper and lower functions for martingales and stationary processes. Ann. Probab. **3** (1975), 119–145.

[13] B. Jamison S. Orey and W. Pruitt, Convergence of weighted averages of independent random variables. Z. Wahrsch. verw. Gebiete **4** (1965), 40-44.

[14] A.N. Kolmogorov, Über das Gesetz des iterierten Logarithmus. Math. Ann. **101** (1929), 126–135.

[15] J. Kubilius, Probabilistic Methods in the Theory of Numbers. Amer. Math. Soc. Translations of Math. Monographs, **11.** Providence, 1964.

[16] J. Marczinkiewicz and A. Zygmund, Remarque sur la loi du logarithme itéré. Fund. Math. **29** (1937), 215–222.

[17] W. Philipp and W. Stout, Invariance principles for martingales and sums of independent random variables. Math. Z. **192** (1986), 253–264.

[18] V. Strassen, Almost sure behavior of sums of independent random variables and martingales. Proc. Fifth Berkeley Symposium Math. Stat. Probab. Vol. II, Part I, 314–343. University of California Press 1965.

[19] M. Weber and M. Lin, Weighted ergodic theorems and strong laws of large numbers. Ergodic Theory Dynam. Systems **27** (2007), 511–543.

[20] M. Weiss, On the law of the iterated logarithm. J. Math. Mech. **8** (1959), 121–132.