LECTURE 4 :

ONE VARIABLE SYSTEMS

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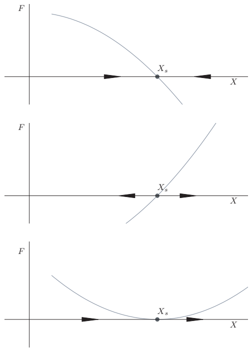
General features

$$\frac{dX}{dt} = F(X, \lambda)$$

- 1d phase space $\rightarrow$ fixed points are only possible attractors
- Real roots ω of characteristic equations $\rightarrow$ monotonic approach towards attractors

$$\omega = \left(\frac{\partial F}{\partial X}\right)_s$$



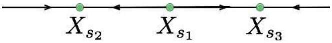
$\omega < 0,\ X_s$ stable

$\omega > 0,\ X_s$ unstable

$\omega = 0,\ X_s$ marginally stable (semi-stable)

General features

- Complexity is here manifested by the coexistence of more than one simultaneously accessible (i.e., stable) steady states ($\equiv$ fixed points).



Transition from single to multiple steady-states ? Relative stability ?

Canonical example from chemical kinetics

Intuitive approach

3d order autocatalysis in an open well-stirred reactor

$$A + 2X \underset{k'}{\overset{k}{\rightleftarrows}} 3X \qquad (A \text{ in excess}) \quad \rightarrow \qquad \frac{dX}{dt} = kAX^2 - k'X^3 + \frac{1}{\tau}(X_0 - X)$$

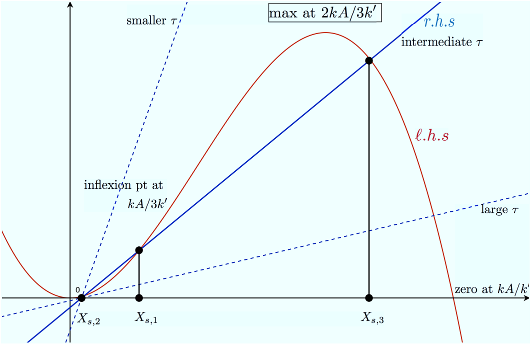
Intuitive approach

$$\frac{dX}{dt} = \underbrace{V(X)}_{\text{production}} + \underbrace{\frac{1}{\tau}(X_0 - X)}_{\text{transport}} \qquad \text{where} \qquad V(X) = kAX^2 - k'X^2$$

Canonical example from chemical kinetics
Intuitive approach

Steady states

$$\underbrace{kAX_s^2 - k'X_s^3}_{\text{l.h.s}} = \underbrace{\frac{1}{\tau}(X_s - X_0)}_{\text{r.h.s}}$$



- Transition between mono and bi stability : r.h.s. tangent to l.h.s.

Canonical example from chemical kinetics
Analytic view

Analytic view

$$\frac{dX}{dt} = kAX^2 - k'X^3 + \frac{1}{\tau}(X_0 - X)$$

4 parameters (too much !). Reduction to two parameters through scaling of X and t.

$$x = \frac{X}{X_0} \qquad T = tk'X_0^2 \qquad \lambda = \frac{kA}{k'X_0} \qquad \mu = \frac{1}{\tau k'X_0^2}$$

$$\Rightarrow \frac{dx}{dT} = -x^3 + \lambda x^2 - \mu x + \mu$$

Canonical example from chemical kinetics
Bifurcation analysis

Bifurcation analysis

$$\frac{dx}{dT} = -x^3 + \lambda x^2 - \mu x + \mu \qquad \text{(after scaling)}$$

- Elimination of x^2 term through transformation $z = x - \frac{\lambda}{3}$

$$\Rightarrow \frac{dz}{dT} = -\left(z + \frac{\lambda}{3}\right)^3 + \lambda\left(z + \frac{\lambda}{3}\right)^2 - \mu\left(z + \frac{\lambda}{3}\right) + \mu$$

or,

$$\Rightarrow \frac{dz}{dT} = -z^3 + \left(\frac{\lambda^2}{3} - \mu\right) z + \left(\frac{2\lambda^3}{27} - \frac{\mu\lambda}{3} + \mu\right) \qquad \text{(I)}$$

Canonical example from chemical kinetics

Bifurcation analysis

- First consider case where constant term vanishes. Condition on μ and λ for this

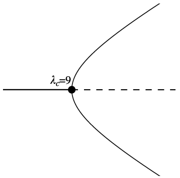
$$\mu = \frac{2\lambda^3}{9(\lambda - 3)} \qquad (\lambda > 3, \text{ since } \mu > 0 \text{ for physical reasons})$$

Eq. for z becomes

$$\frac{dz}{dT} = -z^3 + \frac{\lambda^3 - 9\lambda^2}{9(\lambda - 3)} z \qquad \text{(II)}$$

Steady states :

- $z = 0 \qquad (\lambda > 9)$
- $z_\pm = \frac{\lambda}{3}\sqrt{\frac{\lambda-9}{\lambda-3}} \qquad (\lambda > 9)$
- $z = 0 \qquad (\lambda < 9)$



pitchfork bifurcation

Notice that trivial state $z = 0$ becomes unstable beyond the bifurcation point λ_c. The stability of bifurcating branches can be checked straightforwardly (supercritical bifurcation).

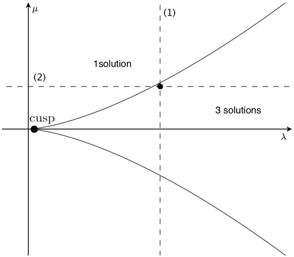
This example is in fact paradigmatic : any system in the vicinity of a pitchfork bifurcation can be reduced to eq. (II) (normal form) where z is a combination of the variables (order parameter). All other variables follow z passively

Canonical example from chemical kinetics
Bifurcation analysis

- In the more general case where the constant term in (I) does not vanish, write equation as

$$\frac{dz}{dT} = -z^3 + \lambda z + \mu$$

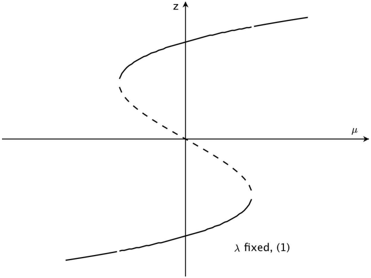
According to the theory of cubic equations, we have the following situation for the steady states :

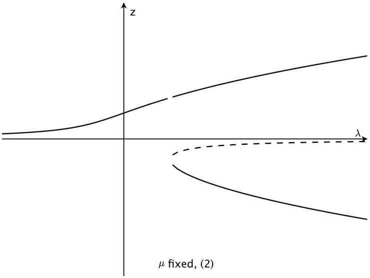


Canonical example from chemical kinetics

Bifurcation analysis

Limit point bifurcations !





Canonical example from chemical kinetics
Kinetic potential and catastrophe theory

A system described by a single variable derives necessarily from a potential, in the sense

$$\frac{dz}{dT} = -\frac{\partial U}{\partial z}$$

For our canonical model, U is obtained by simple quadrature :

$$U = \frac{z^4}{4} - \lambda\frac{z^2}{2} - \mu z$$

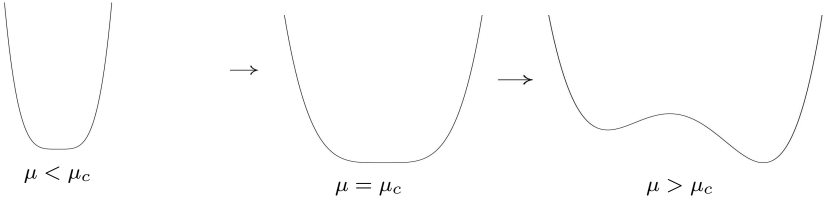
Correspondence to stability :

- z_s stable, U min
- z_s unstable, U max

Canonical example from chemical kinetics

Kinetic potential and catastrophe theory

Transition from one to two stable steady-states



Relative stability :

basins of attraction of the two stable states, or depth of the minimum of the potential.

The concept of structural stability :

classify qualitatively different behaviors that remain robust upon slight changes of the control parameters by determining how the potential is deformed when these parameters are changing.

Canonical example from chemical kinetics

Kinetic potential and catastrophe theory

Catastrophes :

Situations separating qualitatively different behaviors (e.g., cusp point, middle curve of previous slide)

- Full classification possible for cubic nonlinearities as long as two control parameters are available.
- More involved situations for higher order nonlinearities or for multi-variate systems : **catastrophe theory**.

Population dynamics

Verhulst equation

$$\frac{dX}{dt} = \underbrace{kX\left(1 - \frac{X}{N}\right)}_{F(X)}$$

- **Fixed points :**

$$\begin{aligned} X_{s_1} &= 0 \\ X_{S_2} &= N \end{aligned}$$

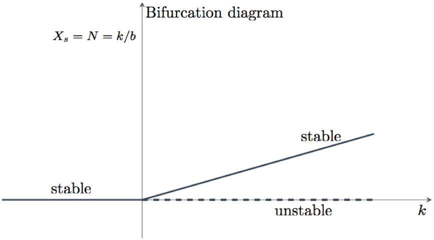
- **Stability :**

$$\begin{aligned} X &= X_s + x \\ \frac{dx}{dt} &= \underbrace{\left(k - \frac{2kX_s}{N}\right)x}_{(\partial F/\partial X)_s \equiv \omega} \end{aligned}$$

Population dynamics

- $X_{s_1} = 0 \Rightarrow \omega = k$
- $X_{s_2} = N \Rightarrow \omega = -k$

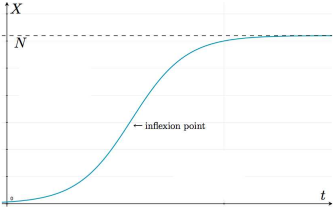
Exchange of stability :



Transcritical bifurcation at criticality $k = 0$.

Analytic solution for $k > 0$:

$$X(t) = N \frac{X(0)}{X(0) + (N - X(0))\, e^{-kt}}$$



Population dynamics

Comparison with data on population growth

Human population :

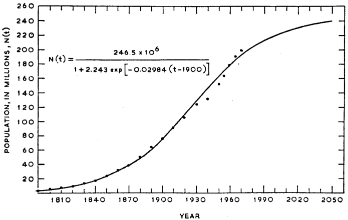


Figure 5 Population of U.S. Logistic curve fitted so that observed points at 1840, 1900 and 1960 are exact.[4] Points represent census data.

Material production :

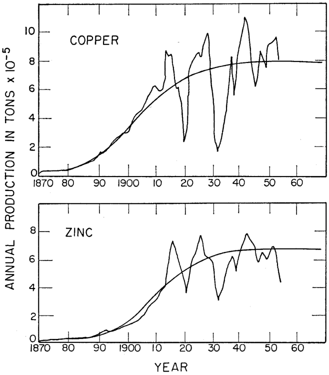
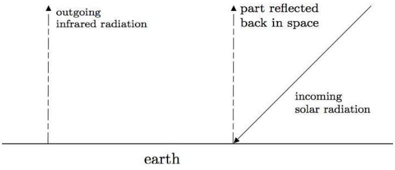


Figure 10 Logistic growth of raw material production, showing oscillation on attaining ceiling conditions (Data from S. G. Lasky, *Eng. Mining J.*, 156 (Sept., 1955).)

Global energy balance and climatic change

Variability of earth's climate over geological time scale. Quaternary glaciations interrupted by interglacial periods.

Does earth's climate admit multiple states ?



Energy balance equation on global scale :

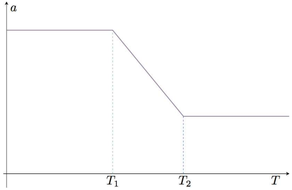
$$C\,\frac{dT}{dt} = \underbrace{Q\left[1 - a\left(T\right)\right]}_{G(T)} - \epsilon\sigma T^4$$

where

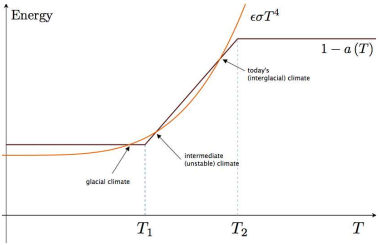
- C : heat capacity
- $G(T)$: incoming minus reflected
- $a(T)$: reflectivity (albedo)
- ϵ CO_2 effect
- σT^4 : Stefan Boltzmann law

Global energy balance and climatic change

Expected form of a : (ice-albedo feedback)



Graphic representation of steady state solutions :



$\Rightarrow$ limit point bifurcation