Lecture 5 :

Two variables

S.C. Nicolis

October - December 2010

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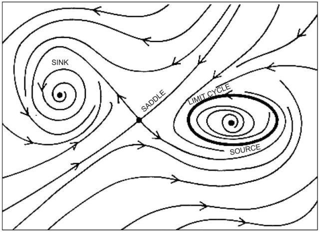
General features of 2-variable systems

$$\frac{dx}{dt} = f(x, y) \qquad \frac{dy}{dt} = g(x, y)$$

- 2-d phase space.
- Characteristic equation for ω of 2$^{\text{nd}}$ degree : possibility of complex conjugate roots and hence of oscillatory behavior.
- Attractors in the form of fixed points and 1-d closed curves (limit cycles)

Typical manifestation of self-organization and complexity.



Main question :

How are multiple steady states and limit cycles born in such systems ?

General features of 2-variable systems

Classification of fixed points

$$x = x_s + \delta x \qquad y = y_s + \delta y$$

$$\begin{aligned} \frac{d\delta x}{dt} &= J_{11}\delta x + J_{12}\delta y \\ \frac{d\delta y}{dt} &= J_{21}\delta x + J_{22}\delta y \qquad (J_{11} = (\partial f/\partial x)_s \text{ etc}) \end{aligned}$$

Characteristic equation :

$$\begin{vmatrix} J_{11} - \omega & J_{12} \\ J_{21} & J_{22} - \omega \end{vmatrix} = 0 \qquad \omega^2 - \underbrace{(J_{11} + J_{22})}_{\text{Trace } T \text{ of } J}\omega + \underbrace{(J_{11}J_{22} - J_{12}J_{21})}_{\text{determinant } \Delta \text{ of } J} = 0$$

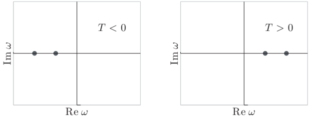
$$\omega_{1,\,2} = \frac{T \pm \left(T^2 - 4\Delta\right)^{1/2}}{2} = \frac{T \pm \mathcal{D}^{1/2}}{2} \qquad (\mathcal{D} = \text{discriminant})$$

General features of 2-variable systems

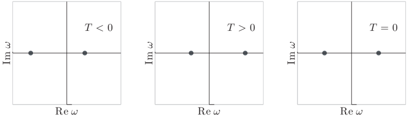
Full list of the different possibilities

$\mathcal{D} > 0$: two real eigenvalues

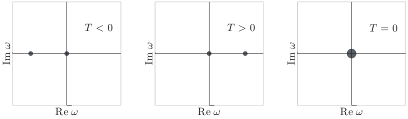
- $\Delta > 0$ roots have the same sign



- $\Delta < 0$ roots have oppositee sign



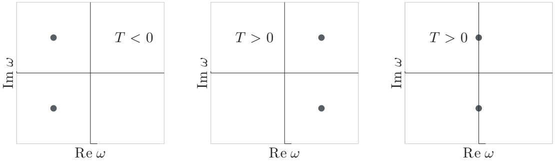
- $\Delta = 0$ At least one of the real roots is zero



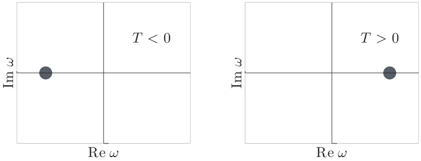
General features of 2-variable systems

Full list of the different possibilities

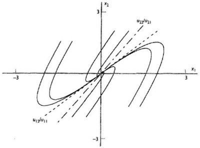
$\mathcal{D} < 0$: two complex conjugate eigenvalues



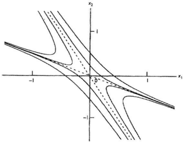
$\mathcal{D} = 0$: double eigenvalue



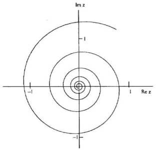
General features of 2-variable systems
Phase portraits



- node, stable case ($\mathcal{D} > 0$, $\Delta > 0$, $T < 0$)



- saddle point ($\mathcal{D} > 0$, $\Delta < 0$)



- focus, stable case ($\mathcal{D} < 0$, $T < 0$)

Limit cycles : Canonical example from chemical kinetics

The irreversible Brusselator

$$A \xrightarrow{k_1} X \qquad B + X \xrightarrow{k_2} Y + C \qquad 2X + Y \xrightarrow{k_3} 3X \qquad X \xrightarrow{k_4} D$$

$$\frac{dX}{dt} = k_1 A - (k_2 B + k_4) X + k_3 X^2 Y$$
$$\frac{dY}{dt} = k_2 B X - k_3 X^2 Y$$

4 parameters (too much !). Again, reduction to 2 parameters through scaling

$$T = k_4 t, \quad x = \left(\frac{k_3}{k_4}\right)^{1/2} X, \quad y = \left(\frac{k_3}{k_4}\right)^{1/2} Y \quad a = \left(\frac{k_1^2 k_3}{k_4^3}\right)^{1/2} A, \quad b = \frac{k_2}{k_4} B$$

$$\Rightarrow \frac{dx}{dT} = a - (b+1)\,x + x^2 y \qquad \frac{dy}{dT} = bx - x^2 y$$

Limit cycles : Canonical example from chemical kinetics

Stationary states

$$x_s = a \qquad y_s = \frac{b}{a}$$

Stability

$$x = a + \delta x \qquad y = \frac{b}{a} + \delta y$$

$$\frac{d}{dt}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} = \overbrace{\begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}}^{\underset{\approx}{J}} \begin{pmatrix}\delta x \\ \delta y\end{pmatrix} + \underbrace{\begin{pmatrix} \frac{b}{a}\delta x^2 + 2a\delta x\delta y + \delta x^2\delta y \\ -\left(\frac{b}{a}\delta x^2 + 2a\delta x\delta y + \delta x^2\delta y\right)\end{pmatrix}}_{\text{nonlinear part}}$$

Limit cycles : Canonical example from chemical kinetics

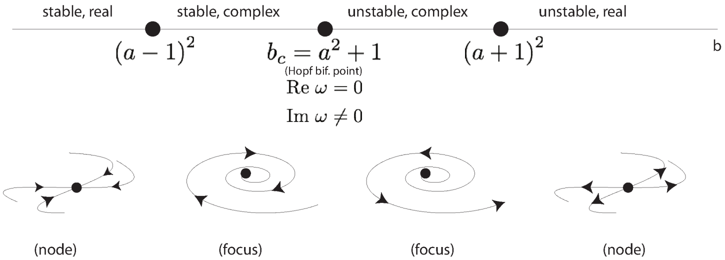
Eigenvalues ω of $\underset{\approx}{J}$ given by the characteristic equation

$$\omega^2 - \left(b - 1 - a^2\right)\omega + a^2 = 0$$

$$\Rightarrow \omega = \frac{b - 1 - a^2 \pm \sqrt{(b - 1 - a^2)^2 - 4a^2}}{2}$$

real	if	$\left(b - 1 - a^2\right)^2 - 4a^2$	> 0
complex	if	$\left(b - 1 - a^2\right)^2 - 4a^2$	< 0
stable	if	$b - 1 - a^2$	< 0
unstable	if	$b - 1 - a^2$	> 0

Limit cycles : Canonical example from chemical kinetics

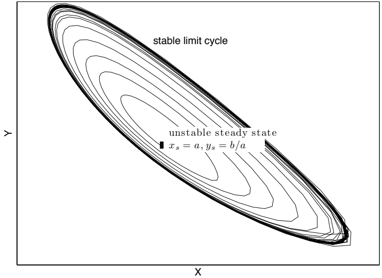


Phase space portraits

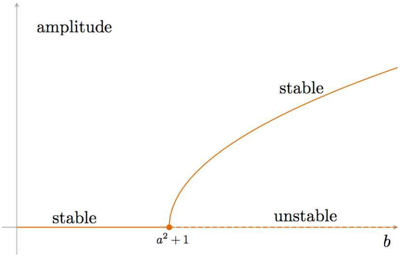
Limit cycles : Canonical example from chemical kinetics

For $b > a^2 + 1$: amplified oscillations of the linearized system

Nonlinearities saturate growth and lead to an **attracting** periodic solution represented by a closed curve in phase space (limit cycle).



Bifurcation diagram



Hopf bifurcation

Some global results related to limit cycles in two variable systems

Bendixson's criterion :

For a closed trajectory to exist, $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ must change sign in the (x, y) plane or vanish identically.

Bendixson's theorem :

The region bounded by a closed trajectory in (x, y) plane contains at least one fixed point (steady state solution)

Poincare-Bendixson's theorem :

Any trajectory staying in a finite region of (x, y) phase space either approaches a fixed point or a periodic orbit. As a corollary, chaotic behavior in continuous time systems can only arise in the presence of at least three coupled variables.

Illustration of Bendixson's criterion

Damped oscillator

x position, v velocity

$$\frac{dx}{dt} = v \equiv f$$
$$\frac{dv}{dt} = -x - \gamma v \equiv g$$

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial v} = -\gamma < 0$$

$\Rightarrow$ **no closed trajectory**

Brusselator

$$f = a - (b+1)\,x + x^2 y$$
$$g = bx - x^2 y$$

Expression changes sign for

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -(b+1) + 2xy - x^2 = 0$$
$$y = \frac{b+1+x^2}{2x}$$

$\Rightarrow$ **possibility of closed trajectory**