Lecture 5b : Spatially extended systems

The Diffusion equation

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Evolution laws in the form of partial differential equation supplemented with appropriate boundary conditions : illustration on the **diffusion equation**

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \qquad 0 \leq x \leq \ell$$

Steady-state solution

$$C_s(x) = \frac{C_2 - C_1}{\ell} x + C_1$$

where $C_1 = C(0)$ and $C_2 = C(\ell)$

Time dependent solution

We choose again the boundary conditions (2) and

$$C(x,0) = C_0(x)$$

as initial condition.
Consider the excess quantity

$$u(x,t) = C(x,t) - C_s(x)$$

u then satisfies

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

with

$$\begin{aligned} u(0) &= u(\ell) = 0 \\ u(x,0) &= C_0 - C_s(x) \\ &\equiv u_0(x) \end{aligned}$$

Time dependent solution

ϕ_m :eigenfunctions of the diffusion operator $\partial^2/\partial x^2$ (Diffusion operator is **dissipative**) :

$$\frac{d^2\phi_m(x)}{dx^2} = -k_m^2\phi_m(x)$$

Solutions in the form of an infinite series of ϕ_m's

$$u(x,t) = \sum_m A_m(t)\,\phi_m(x)$$

fix the $A_m(0)$'s by requiring that

$$\sum_m A_m(0)\,\phi_m(x) = u(x,0) = u_0(x)$$

leading to

$$A_m(0) = \frac{\int_0^\ell dx\phi_m^*(x)u_0(x)}{\int_0^\ell dx\phi_m^*(x)\phi_m(x)}$$

Time dependent solution

and, finally

$$u(x,t) = \sum_m \frac{\int_0^\ell dx \phi_m^*(x) u_0(x)}{\int_0^\ell dx \phi_m^*(x) \phi_m(x)} \mathrm{e}^{-Dk_m^2 t} \phi_m(x)$$

or, by computing ϕ_m and k_m explicitely

$$u(x,t) = \sum_m \frac{2}{\ell} \left(\int_o^\ell dx \sin \frac{m\pi x}{\ell} u_0(x) \right) \sin \frac{m\pi x}{\ell} \mathrm{e}^{-D \frac{m^2 \pi^2}{\ell^2} t}$$

with m integer