Lecture 6 : Spatially extended systems

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Evolution laws in the form of partial differential equation supplemented with appropriate boundary conditions

Principal new feature : pattern formation

Spontaneous onset of solutions exhibiting a space dependence that is qualitatively different from that of the system's geometry and of the external environment (spontaneous symmetry breaking).

Two representative cases :

- Turing instability (pattern formation concomitant to the loss of stability of the uniform state), illustrated on the **Brusselator**.
- Wave propagation, illustrated on the **Fisher equation**.

Linear stability analysis in the presence of diffusion

$$X_i\,(r,t) = \underbrace{X_{is}}_{\text{uniform steady state}} + \underbrace{x_i(r,t)}_{\text{perturbation}}$$

Linearized reaction-diffusion equations :

$$\frac{\partial x_i}{\partial t} = \sum_j J_{i,j} x_j + D_i \nabla^2 x_i \qquad (1)$$

+ boundary conditions

We introduce the eigenfunctions and eigenvalues of ∇^2 compatible with the boundary conditions,

$$\nabla^2 \phi_m(r) = -k_m^2 \phi_m(r) \qquad (2)$$

Linear stability analysis in the presence of diffusion

and seek for solutions of (1) in the form

$$x_i = u_i \mathrm{e}^{\omega t} \phi_m(r) \tag{3}$$

(justified from the fact that the coefficients in (1) are constant since they are evaluated at the uniform steady state)

We obtain (after simplifying by $\mathrm{e}^{\omega t}\phi_m(r)$)

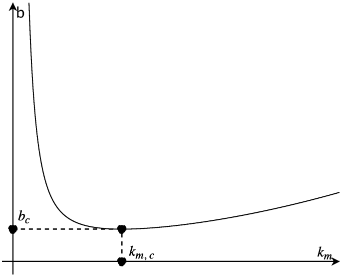
$$\sum_j J_{ij} u_j - \left(D_i k_m^2 + \omega\right) u_i = 0$$

or,

$$\sum_j \left[J_{ij} - \left(D_i k_m^2 + \omega\right) \delta_{ij}^{\mathsf{kr}}\right] u_j = 0$$

Linear stability analysis in the presence of diffusion

leading to the characteristic equation determined by the calculation of the determinant. The onset of a symmetry-breaking instability will be signaled by $\omega_c = 0$ for a critical parameter value (say, b) hidden in the coefficients J_{ij}. The characteristic equation provides a relation linking b to k_m. If this relation is of the form



then at the instability threshdold b_c the solutions (3) will have a non-trivial space dependence, since $k_{m_c} \neq 0$. This provides us a quantitative criterion for checking the possibility of a symmetry breaking (Turing) instability.

Linear stability analysis in the presence of diffusion

Illustration on the Brusselator

$$\begin{aligned} \frac{\partial x}{\partial T} &= a - (b+1)\,x + x^2 y + D_1 \nabla^2 x \\ \frac{\partial y}{\partial T} &= bx - x^2 y + D_2 \nabla^2 y \end{aligned}$$

Seek for solutions in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \phi(\underline{r}) \mathrm{e}^{\omega T}$$

where $\phi(\underline{r})$ is an eigenfunction of the Laplacian ∇^2

$$\nabla^2 \phi_m = -k_m^2 \phi_m \qquad (\text{e.g. } \phi_m \approx \mathrm{e}^{i k_m \underline{r}} \text{ for periodic boundary conditions})$$

Characteristic equation of $\underset{\approx}{J}$ becomes

$$\omega^2 - \left(b - a^2 - 1 - (D_1 + D_2)\,k_m^2\right)\omega + a^2 + \left(a^2 D_1 k_m^2 - (b-1)\,D_2 k_m^2\right) + D_1 D_2 k_m^4 = 0$$

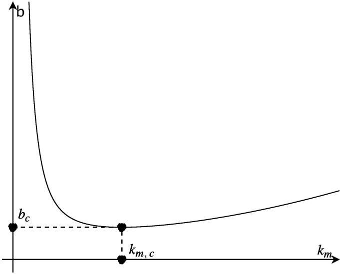
Linear stability analysis in the presence of diffusion

Illustration on the Brusselator

Criticality possible for real ω's.

$\omega = 0$ for

$$b = D_1 k_m^2 + a^2 \frac{D_1}{D_2} + 1 + \frac{a^2}{D_2 k_m^2}$$



minimum at

$$b_c = \left(1 + \sqrt{\frac{D_1}{D_2}} a\right)^2$$

$$\underbrace{k_{m,c}^2 = \frac{a}{(D_1 D_2)^{1/2}}}_{\text{non-trivial space dependence!}}$$

$\Rightarrow$ TURING INSTABILITY : prelude to pattern formation

Linear stability analysis in the presence of diffusion

Illustration on the Brusselator

Condition for Turing instability

Condition for Turing instability to take over the instability leading to limit cycle behavior :

$$
\begin{aligned}
b_c(\text{Turing}) &< b_c(\text{Hopf}) \qquad \text{or} \\
\left(1+\sqrt{\frac{D_1}{D_2}}a\right)^2 &< 1+a^2
\end{aligned}
$$

$\Rightarrow \quad D_1 < D_2$

Experimental evidence :

Belousov-Zhabotinski reaction, ants cemeteries (*cf.* Lecture 1).

Wave propagation

Generalities

Traveling wave : a disturbance propagating at finite (constant) velocity in space :

$$X(r,t) = f(r - vt)$$

where $\underline{v}$ is the propagation velocity

Ubiquity of traveling waves in nature :

- linear waves
  - electromagnetic waves (at basis of telecommunications)
  - sound waves (at basis of everyday communication)
- nonlinear waves
  - water waves (at basis of oceanic circulation)
  - nerve impulse (at basis of cognition)
  - propagation of innovations (mutations, rumors, ...)

Wave propagation

Typical mechanism for nonlinear wave generation in a reaction-diffusion system :

$$\frac{\partial X_i}{\partial t} = F_i \left(X_1, ...X_n \right) + D \nabla^2 X_i$$

X_{is} fixed points, solutions of $F_i \left(\{ X_{js} \} \right) = 0$.

Let there be several fixed points. A propagating wave can then exist, in principle, as a phase space trajectory joining pairs of fixed points.

Wave propagation

Illustration : Fisher's equation in 1-d space

$$\frac{\partial X}{\partial t} = kX\left(1 - \frac{X}{N}\right) + D\frac{\partial^2 X}{\partial r^2} \qquad (-\infty < r < \infty)$$

First scaling

$$\begin{aligned} x &= \frac{X}{N} \quad \Rightarrow \\ \frac{\partial x}{\partial t} &= kx\,(1-x) + D\frac{\partial^2 x}{\partial r^2} \end{aligned}$$

Second scaling

$$\tau = kt, \qquad \rho = r\,(k/D)^{1/2}$$

Equation becomes,

$$\frac{\partial x}{\partial \tau} = x\,(1-x) + \frac{\partial^2 x}{\partial \rho^2}$$

Wave propagation

Illustration : Fisher's equation in 1-d space

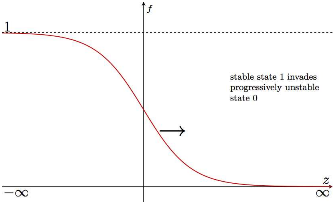
Traveling wave :

$$x = f\left(\underbrace{\rho - v\tau}_{z}\right)$$

(*) transformed in an o.d.e for f

$$f'' + vf' + f(1-f) = 0 \qquad (**)$$

where differentiation is with respect to z.
Expected shape of f :



i.e., $f(z=-\infty)=1,\ f(z=\infty)=0.$

It can be shown by a phase space analysis that solutions of the above kind exist for any $v \geq 2$ (or, in initial variables, $v \geq 2\sqrt{kD}$).
Furthermore, for sufficiently sharply varying initial conditions, all solutions tend to the wave associated to the minimum speed $v_{\min} = 2$.

Wave propagation

Illustration : Fisher's equation in 1-d space

Analytic construction :

new scaling : $\xi = \frac{z}{v} = \epsilon^{1/2} z$, $f = g(\xi)$. where $\epsilon = 1/v^2$ is regarded as a small quantity ($\epsilon \leq 0.25$).
(**) becomes

$$\epsilon \frac{d^2 g}{d\xi^2} + \frac{dg}{d\xi} + g(1-g)$$

Seek for solutions

$$y = g_0(\xi) + \epsilon g_1(\xi) + \ldots$$

Then, to the dominant order in ϵ,

$$\frac{dg_0}{d\xi} = -g_0(1-g_0) \qquad \Rightarrow \qquad g_0(\xi) = \frac{1}{1+\mathrm{e}^{\xi}}$$

or, in original variables,

$$f(z) = \frac{1}{1+\mathrm{e}^{\mathrm{z}/\mathrm{v}}} \qquad + \text{corrections of higher order in } 1/v^2$$