Project I

Reading:

Strogatz, Nonlinear Dynamics and Chaos, Chapter 6
Edelstein-Keshet, Mathematical Models in Biology, Chapter 6

Some Papers You can download from the webpage

1 A linear oscillator

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

transformed to two variables through

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\omega^2 x \end{aligned}$$

1. *(1/20)* For $v_0 = 0$ and $x_0 = 0$, 0.1 and 0.2 as initial conditions and $\omega \neq 0$, integrate numerically the equations, draw the phase space portrait and identify the invariant sets.

2 The Lotka-Volterra oscillator

Consider a population density x of rabbits (the preys) and a population density y of foxes (the predators). Their interactions can be formulated in the following way:

$$\frac{dx}{dt} = ax - kxy$$
$$\frac{dy}{dt} = kxy - \gamma y$$

where a is the birth rate of rabbit's population, k the interaction rate between rabbits and foxes and γ is the death rate of the fox population

1. *(3/20)* For the parameter values of your choice (don't forget though that parameters are *rate parameters*) and initial conditions equal to $\{x_0 = 0.1, y_0 = 0.1\}$ and $\{x_0 = 1, y_0 = 1\}$, integrate numerically the equations, draw the phase space portrait and identify the invariant sets.

2. *(1/20)* Discuss the biological meaning of the invariant sets

3. *(2/20)* Find a chemical equivalent scheme

4. *(3/20)* Show that the quantity $U = k(x+y) - \gamma \ln x - a \ln y$ is conserved during the dynamics.

3 The Oregonator

The Oregonator model captures the essential features of the **Belousov-Zhabotinsky** reaction. The kinetic scheme can be written as

$$\begin{aligned} A + Y &\xrightarrow{k_1} X \\ X + Y &\xrightarrow{k_2} P \\ B + X &\xrightarrow{k_{34}} 2X + Z \\ X + X &\xrightarrow{k_5} Q \\ Z &\xrightarrow{k_6} Y \end{aligned}$$

1. *(1/20)* Specify from the litterature the chemical equivalents of X, Y and Z.

2. *(2/20)* Write the balance equations for X,Y and Z

3. *(3/20)* It can be shown that the balance equantions can be scaled in the following way

$$
\begin{aligned}
\frac{1}{\epsilon}\left(\frac{dx}{d\tau}\right) &= y - xy + x\left(1 - qx\right) \\
\epsilon\left(\frac{dy}{d\tau}\right) &= -y - xy + z \\
\frac{dz}{d\tau} &= w\left(x - z\right)
\end{aligned}
$$

Integrate numerically the equations and draw phase portrait in projection into x, y plane and in a 3-d perspective for parameter values equal to $\epsilon = 77.27$, $q = 8.375 \times 10^{-6}$ and $w = 0.1610$ and different initial conditions. Identify the invariant set.

4 The Lorenz model of thermal convection

$$
\begin{aligned}
\frac{dx}{dt} &= \sigma\left(-x + y\right) \\
\frac{dy}{dt} &= rx - y - xz \\
\frac{dz}{dt} &= xy - bz
\end{aligned}
$$

For $\sigma = 10$, $b = \frac{8}{3}$ and $r = \frac{1}{2}$, 2, 23.5 and 28 and initial conditions for x, y and z between $[-1 : 1]$:

- *(3/20)* Draw phase portrait in projection into x, z plane and in a 3-d perspective
- *(1/20)* Comment on the qualitative changes as r is gradually increased.