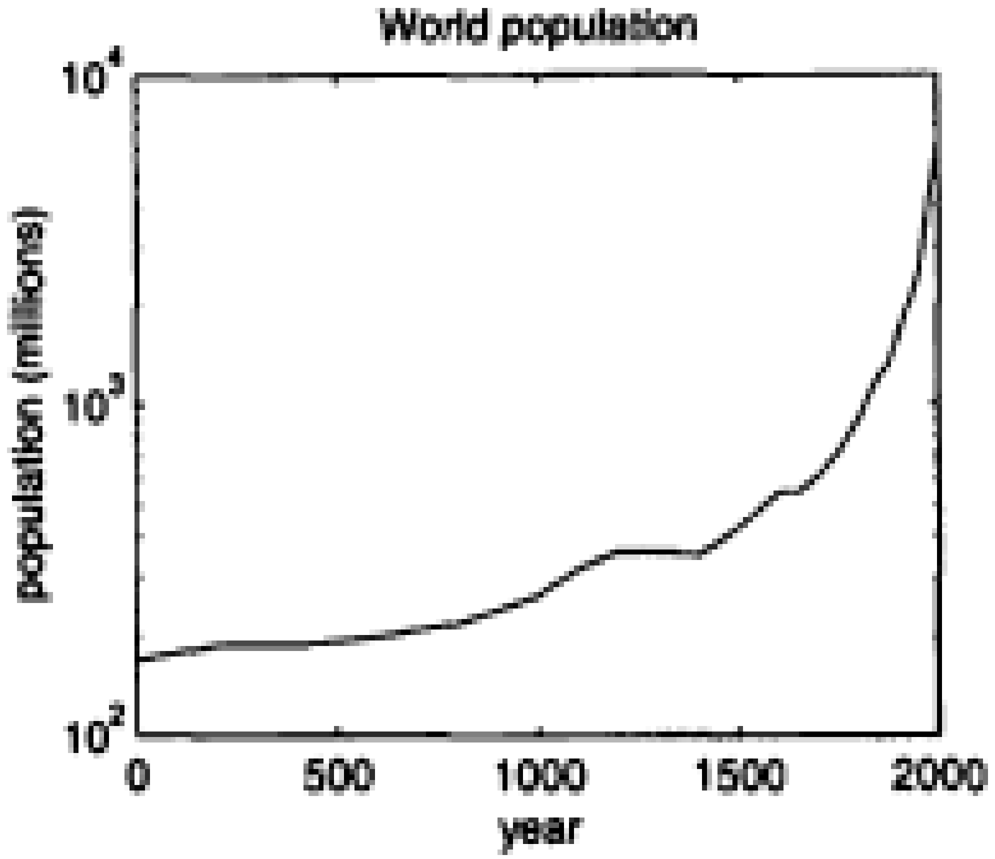
Mathematical Biology - **Lecture 3 –** population

and then there were many

- A **population** is all the organisms of the same group or species who live in the same geographical are and are capable of interbreeding.
- population studies – censuses from Roman times, more elaborate modern versions, birdwatchers, pugmarks – rich statistics
- Fibonacci – one of the first models of population – rabbits that don’t die
- Euler, Laplace
- models that explain the data through self-regulating mechanisms
- models that look explicitly at interactions between species and environment

the population according to Malthus

dynamics of population – how does the population change over time



generations – discrete-time or metered models

Thomas Robert Malthus – exponential growth in population limited only be famine, disease etc

in discrete time, $N_{n+1} = (1 + b - d)N_n = \lambda N$

in continuous time, $\frac{dN}{dt} = rN$

beetles, bud moths and blowflies

useful in modeling insect populations etc when intra-specific competition for resources is the critical factor

$N_{n+1} = R_0 S(N_n) N_n = f(N_n)$, R_0 - average number of offspring, S() – survival function

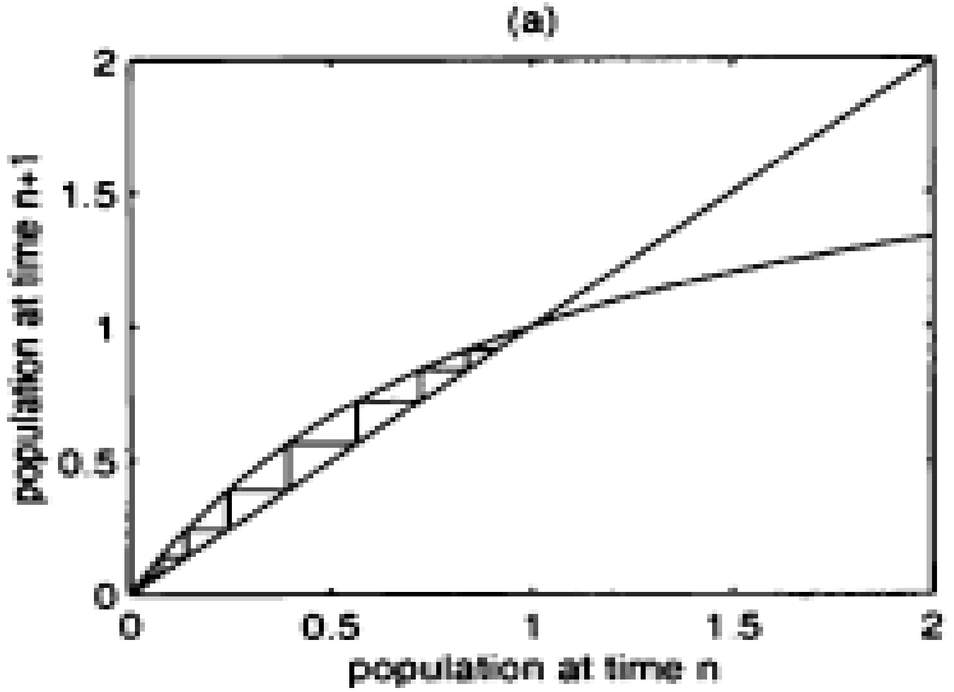
Contest competition – winner takes all

Scramble competition – equal shares

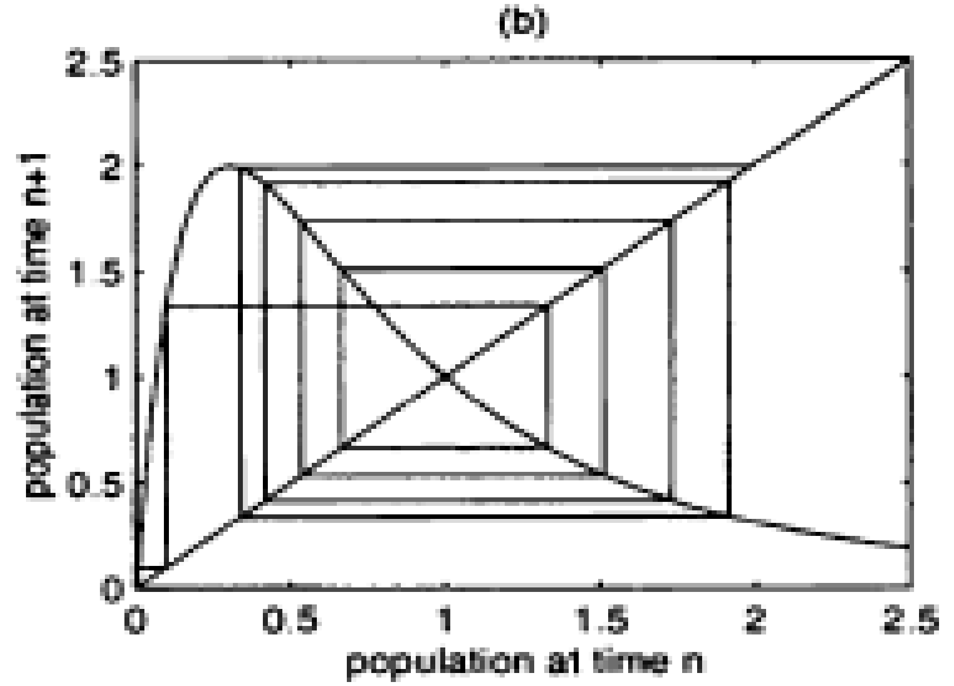
in real data, we see over-compensation, under-compensation not perfect compensation

Hassell equation: $N_{n+1} = f(N_n) = \frac{R_0 N_n}{(1+aN_n)^b}$

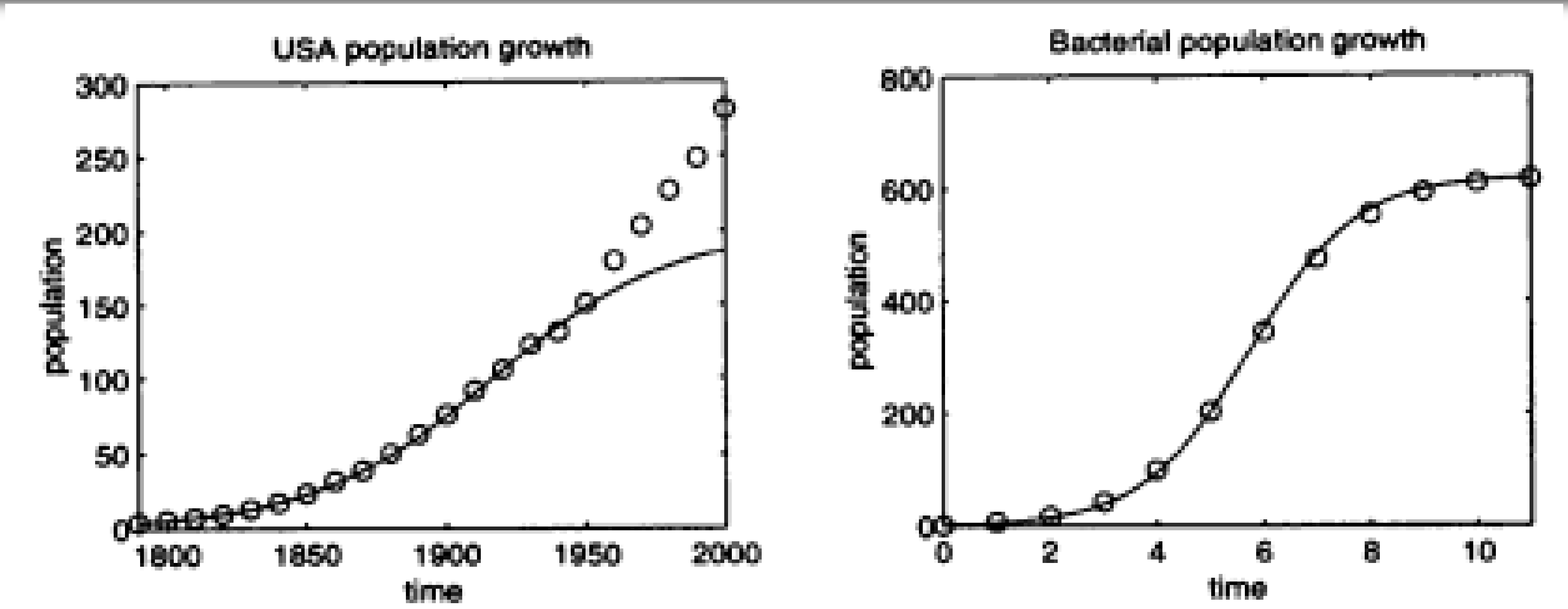
b = 1



b > 1



the population according to Verhulst



How do the limiting factors to population work?

Malthus: $\frac{dN}{dt} = f(N) = (b-d)N = \ rN$

Verhulst: $\frac{dN}{dt} = f(N) = rN(1-\frac{N}{K})$, quadratic term inspired from physics

r – net per capita growth rate as before, K – carrying capacity of the environment

how many people on earth

Malthus: $N(t) = N_0 e^{rt}$

Verhulst: $N(t) = \frac{N_0 e^{rt}}{K - N_0 + N_0 e^{rt}}$

the Malthusian model is the simplest and is often used when a population model has to be embedded in more complex models

the logistic equation has been successful in explaining many populations or related effects
Earth's carrying capacity:2 billion in 1924, revised to 2.6 billion in 1936

Allee effect – depensatory growth – guillemots

what do we choose: K or r

$$\frac{dN_1}{dt} = r_1 N_1 (1 - \frac{N}{K_1}), \frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N}{K_2}\right), N = N_1 + N_2$$

A mutant competing with the original population – but does it invade?

$(K_1, 0)$ is a steady state but if it is not stable, we can say the mutant invades.

The Jacobian matrix is given by $\begin{bmatrix} -r_1 & -r_1 \\ 0 & r_2 \left(1 - \frac{K_1}{K_2}\right) \end{bmatrix}$

age-structure

Fibonacci rabbits:

$$\begin{matrix} z_{1,n+1} \\ z_{2,n+1} \end{matrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{matrix} z_{1,n} \\ z_{2,n} \end{matrix}$$

Leslie matrices: $z_{n+1} = L z_n$

$$L = \begin{bmatrix} s_1 m_1 & s_1 m_2 \cdots & s_1 m_{\omega-1} & s_1 m_\omega \\ & 0 & 0 & 0 \\ s_2 & s_3 \ddots & \vdots & \\ 0 & \cdots & s_\omega & 0 \\ 0 & & & \end{bmatrix}$$

s_i – survival function – probability of surviving from age i-1 to i

m_i- maternity function at age i

Euler-Lotka equations

interacting species

population of any one species depends on interactions with other species

competition – inhibitory effect for both
symbiosis or mutualism – beneficial effect for both
predation or parasitism – opposite effects for prey and predator

we look at predation: host-parasitoid interactions

Nicholson-Bailey model:
non-overlapping generations of parasitoids
parasitised host dies

Nicholson-Bailey

H_n, P_n- number of hosts, parasitoids at generation n
R_0 - basic reproductive ratio of host
c – average number of parasite eggs that survive to breed
f(H,P) – fraction not parasitised

Census takes place at the beginning of season before parasitism

$$H_{n+1} = R_0 H_n f(H_n, P_n), \qquad P_{n+1} = cH_n(1 - f(H_n, P_n))$$

Jacobian at steady state: $\begin{bmatrix} R_0(f^* + H^* f_H^*) & R_0 H^* f_P^* \\ c(1 - f^* - H^* f_H^*) & -cH^* f_P^* \end{bmatrix}$

Jury conditions for stability: |tr(J)| < det(J) + 1, det(J) < 1

Nicholson-Bailey

Nicholson-Bailey assumes parasitoids search for hosts according to a Poisson process with parameter a

$$f(H_n, P_n) = \exp(-aP_n)$$

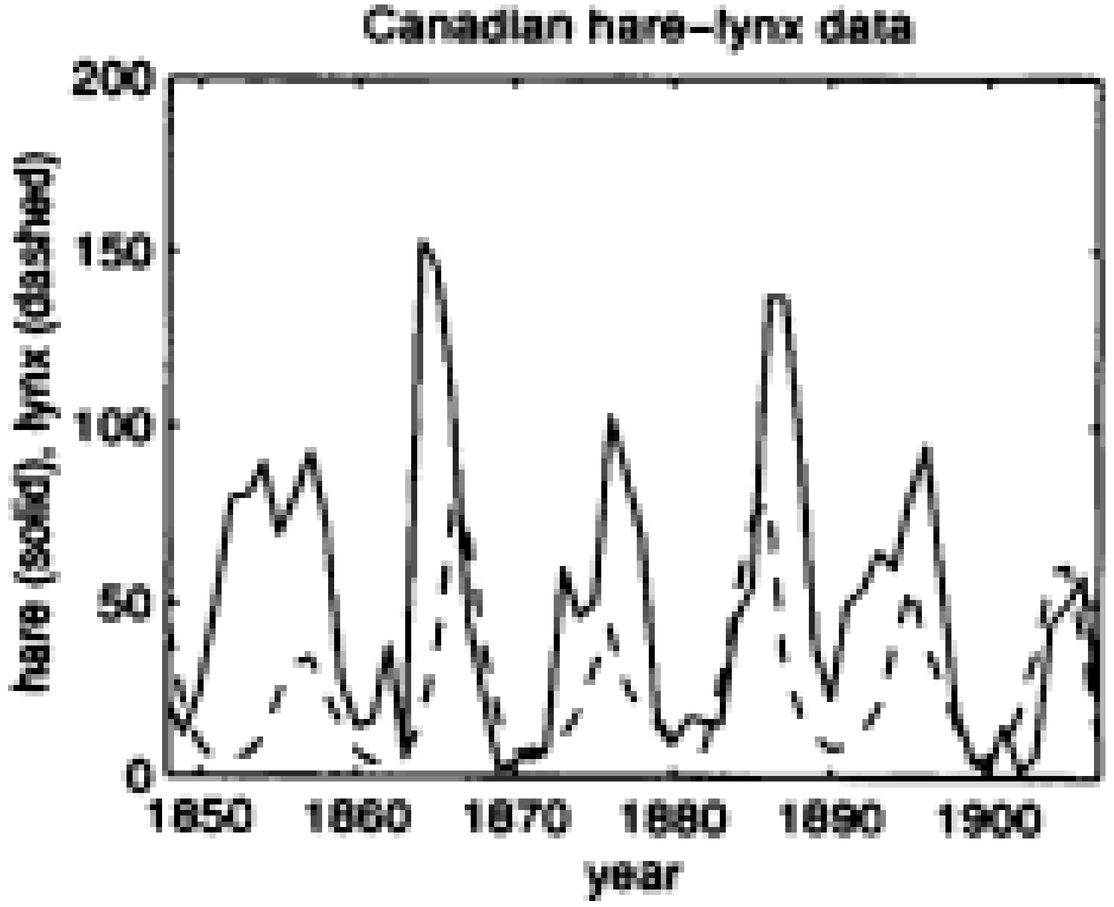
Justification: Each season the parasitoids search for hosts randomly and the number of hosts changes as $\frac{dH}{dt} = -\alpha PH$

assuming parasitoid population is constant and integrating over the time of the search, $\mathrm{H}(\mathrm{n} + \tau) = H_n \exp(-\alpha P_n \tau)$

Lotka-Volterra

Modeling predator-prey interactions – fishing in the Mediterranean – Volterra and his son-in-law, independently Lotka

Laws of theoretical ecology



Lotka-Volterra - assumptions

U – number of prey, V – number of predators

rate of change of U = net rate of growth without predation – loss due to predation
rate of change of V = net rate of growth due to predation – loss without prey

- Prey limited only by predator, otherwise grows exponentially
- Predation term linear in U
- No interference between predators in finding prey
- Without prey, predator dies off exponentially
- Every unit of prey death contributes to unit growth in predator

Lotka-Volterra - equations

$$\frac{dU}{d\tau} = \alpha U \ - \gamma UV, \frac{dV}{d\tau} = e\gamma UV \ - \beta V$$

Steady state at (0,0)

Non-trivial steady state at $(\frac{\beta}{e\gamma}, \frac{\alpha}{\gamma})$

In the Volterra fishing example, we can add catchability coefficients for predator and prey p, q and constant effort E:

$$\frac{dU}{d\tau} = \alpha U \ - pEU - \ \gamma UV, \frac{dV}{d\tau} = e\gamma UV \ - qEV - \ \beta V \text{ etc}$$

Lotka-Volterra - analysis

Non-dimensionalising by u = U/U*, v = V/V* and rescaling time, we get

$$\frac{du}{dt} = u(1 - v), \qquad \frac{dv}{dt} = av(u - 1), a = \frac{qE + \beta}{\alpha - pE}$$

We get the equation of the phase plane as

$$\frac{dv}{du} = \frac{av(u - 1)}{u(1 - v)}$$

This has periodic solutions:

$$\Phi(u, v) = a(u - \log u) + v - \log v = A$$

Lotka-Volterra - analysis

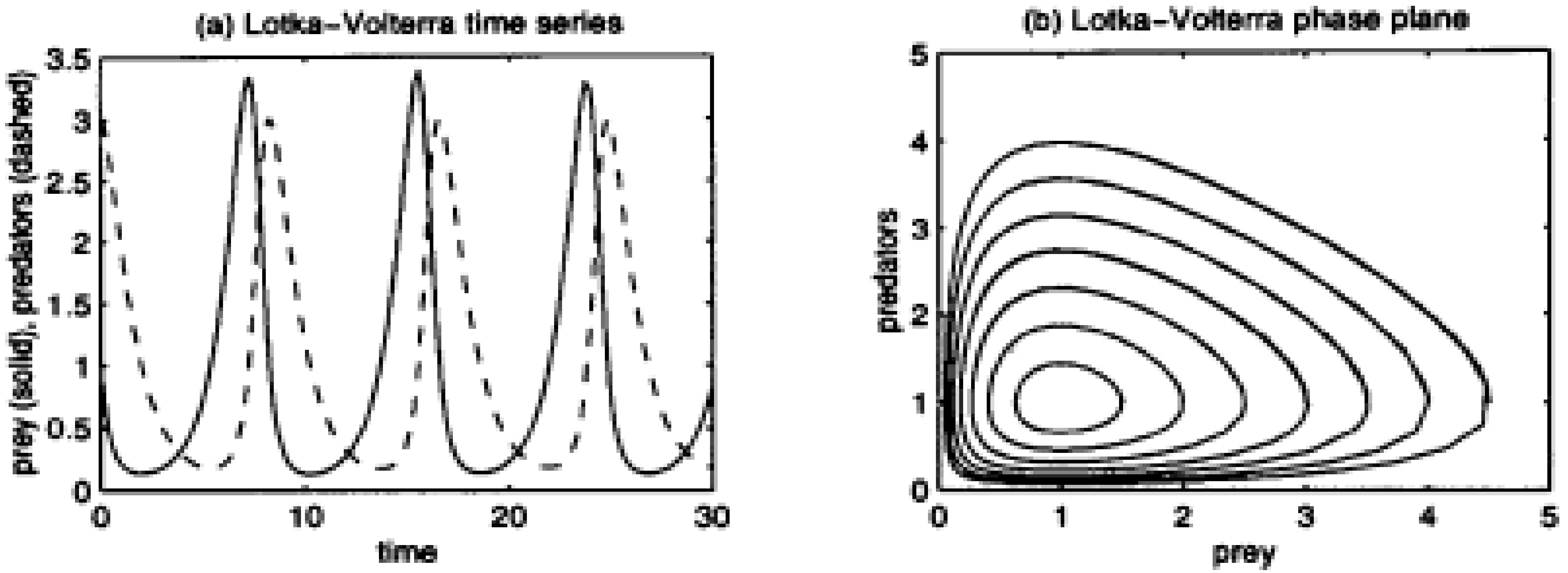


Figure 2.6 Some numerical solutions of the non-dimensional Lotka–Volterra prey-predator model Equations (2.3.9) and the corresponding phase plane Equation (2.3.10).

Average population is the steady state population but both prey and predator populations crash in every cycle