Mathematical Biology - Lecture 5 – Turing patterns

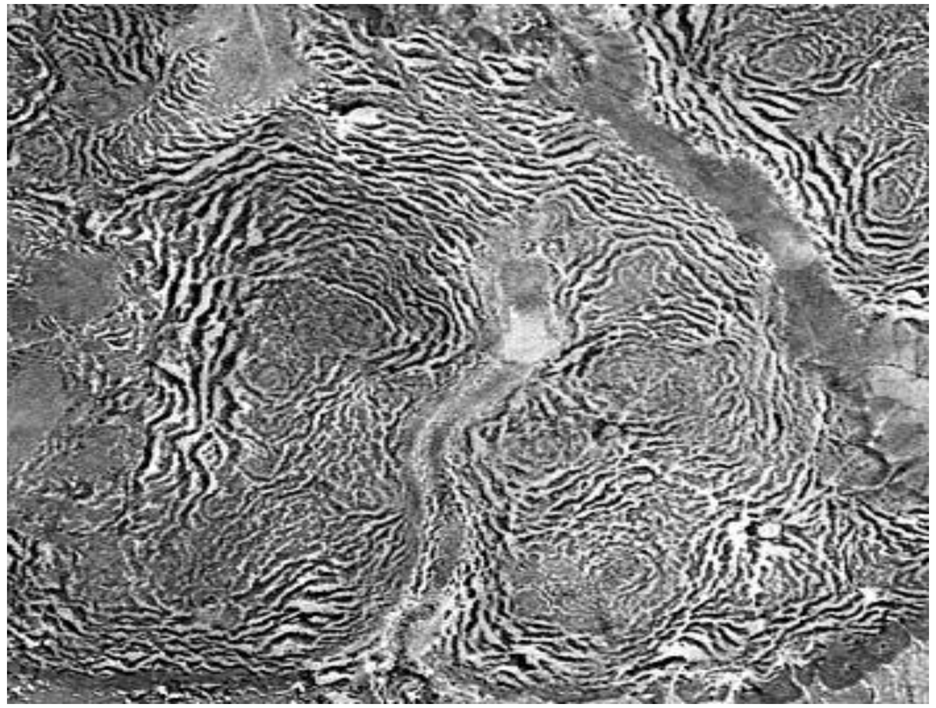
Turing patterns

complex organisms form from a simple egg – asymmetric structures

patterns in nature – spots, stripes, vegetation – can they form spontaneously?

Alan Turing – The Chemical Basis of Morphogenesis

symmetry.-breaking by short-range activation and long-range inhibition



diffusion equation

$$\frac{\partial u}{\partial t} = \alpha f(u) + D\nabla^2 u$$

homogeneous Neumann boundary conditions

linearised equation with perturbations:

$$\frac{\partial v}{\partial t} = \alpha J^* v + D\nabla^2 v$$

find spatial eigen values λ and eigen functions $F_n(x)$
find temporal eigen values and eigen vector for the matrix
$A = \alpha J^* - \lambda D$
general solution $v(x,t) = \sum_{n=0}^{\infty} \sum_{i=1}^{m} a_{ni} c_{ni}(t) F_n(x) \exp(\sigma_{ni} t)$

analysis

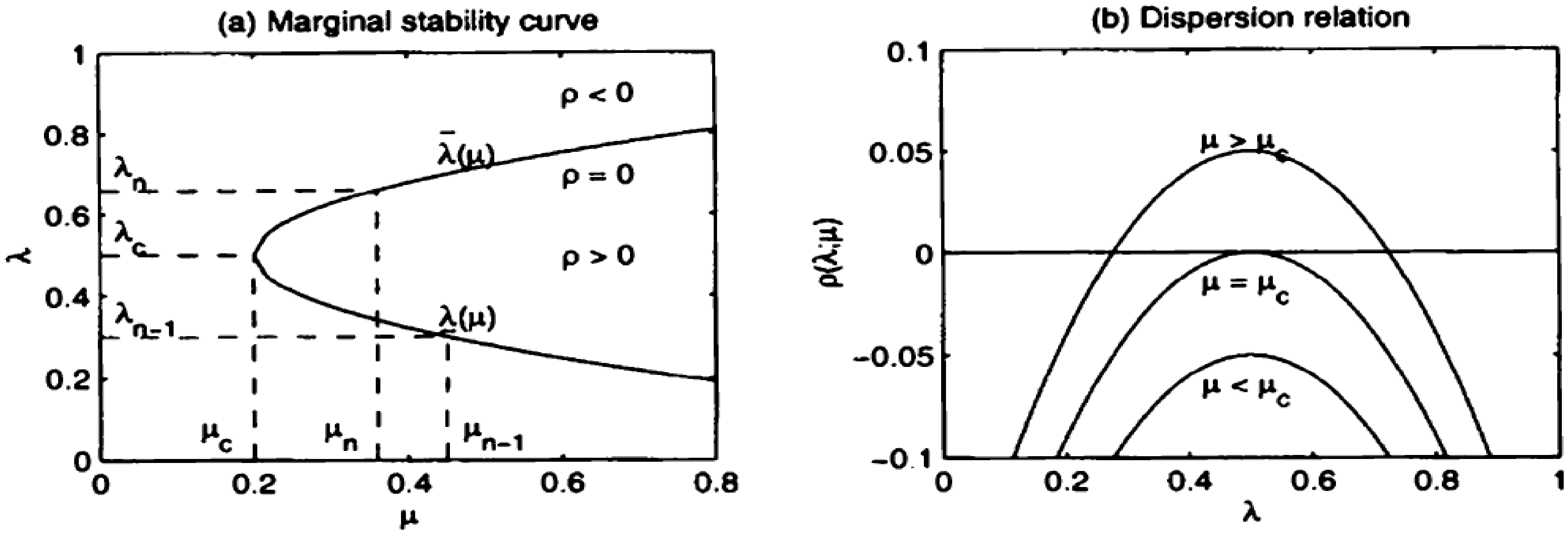
suppose the system is stable to spatially homogeneous perturbations, ie., for spatial eigen value 0 but not for other spatial eigen values

since $\det(\sigma I - A) = \det(\sigma I - \alpha J^* + \lambda D) = 0$, by definition, for each spatial eigen value, there are m roots for σ

$\rho(\lambda) = \max_i Re\ \sigma_i(\lambda)$ - dispersion relation

assume all paramters but one are fixed and investigate the dispersion relation to find critical values

analysis



$$\rho(\lambda_c;\mu_c) = \frac{d\rho}{d\lambda}(\lambda_c,\mu_c) = 0$$

can write as $Q(\sigma) = \sigma^m + a_1(\lambda;\mu)\sigma^{m-1} + \cdots + a_m(\lambda;\mu) = 0$

and at the critical point, $a_m(\lambda;\mu) = \frac{da_m}{d\lambda} = 0$

activator-inhibitor systems

with only one dimension, diffusion is stabilising, so Turing instability needs at least 2 dimensions – analyse system of 2 reaction-diffusion equations

look at the Jacobian – Stability to homogeneous perturbation implies

$$f_u^* + g_v^* < 0, f_u^* g_v^* - f_v^* g_u^* > 0$$

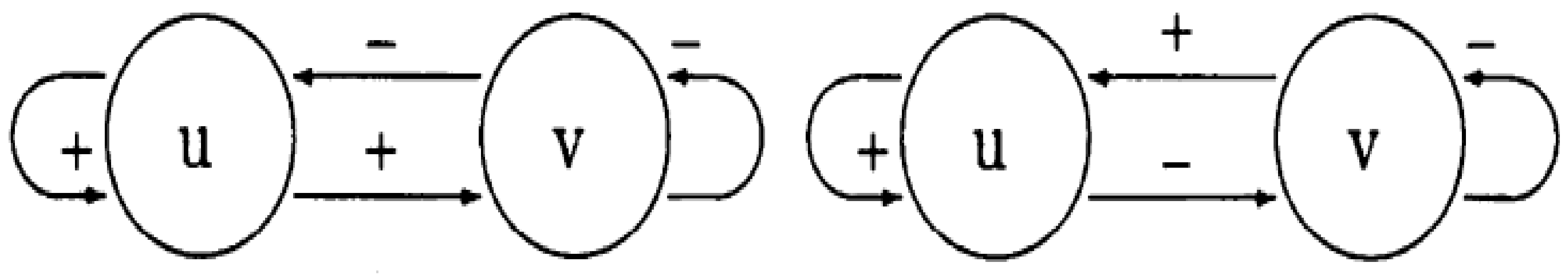
Critical point for reaction-diffusion process is at

$$\lambda_c = \frac{1}{2}\alpha\left(\frac{f_u^*}{D_1} + \frac{g_v^*}{D_2}\right)$$

activator-inhibitor systems

Using the condition on stability for homogeneous perturbation and the fact that critical eigen value >0, we have an activation-inhibition setup

$$\begin{pmatrix} + & - \\ + & - \end{pmatrix} \text{ or } \begin{pmatrix} + & + \\ - & - \end{pmatrix}$$



range of u: $\mathrm{r}_1 = \sqrt{\frac{2D_1}{\alpha f_u^*}}$, range of v: $\mathrm{r}_2 = \sqrt{\frac{2D_2}{\alpha |g_v^*|}}$, $\quad r_2 > r_1$