Summer school on collective behaviour

Differential equations models

Dynamical systems governed by ordinary differential equations

1. Geometric view

2. Linear stability analysis

3. Example

Evolution of a system described by a set of Ordinary Differential Equations (O.D.E.)

$$\frac{dX_i}{dt} = F_i\left(X_1, ..X_n, \lambda\right)$$

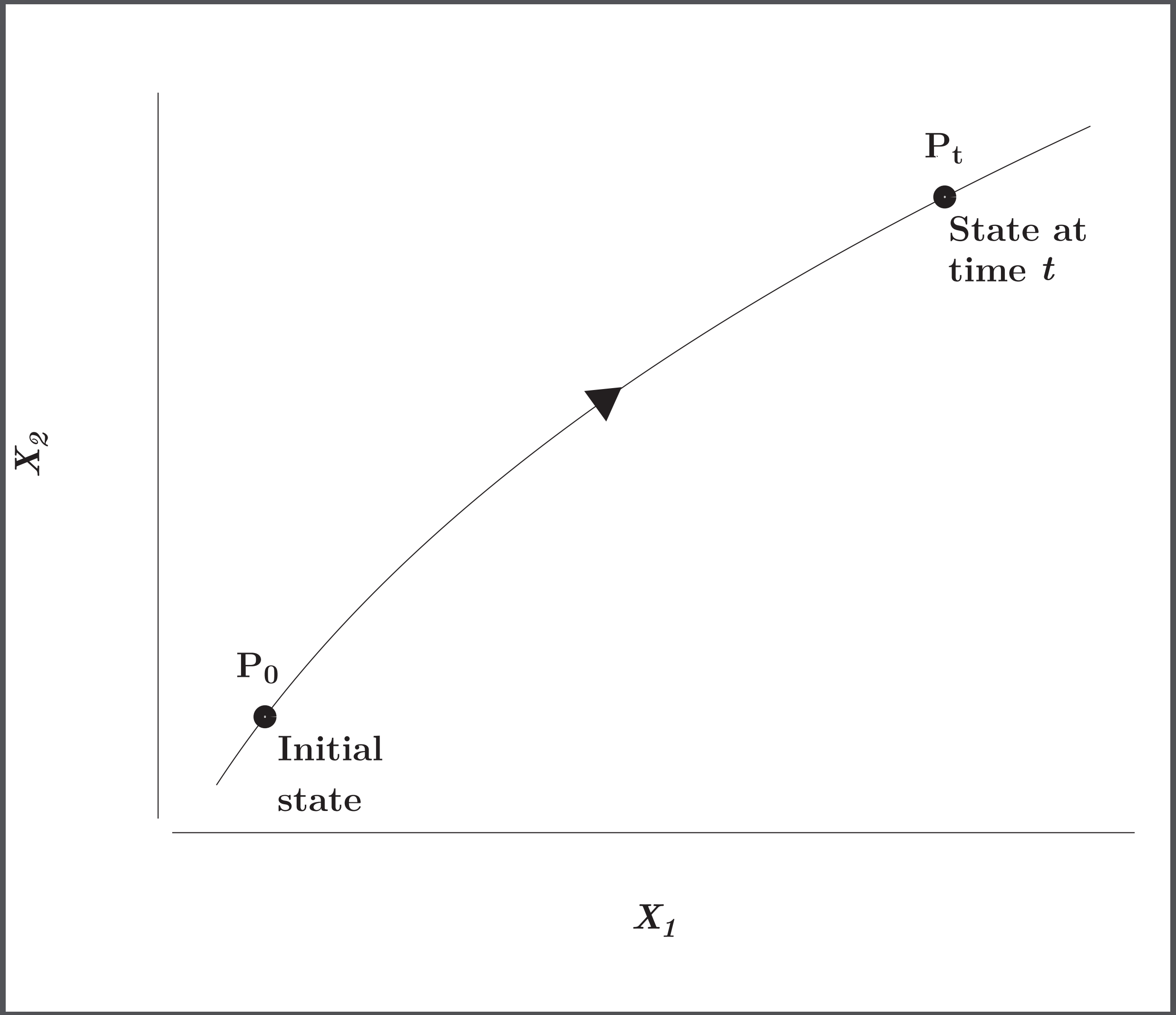
where

- X_i are the variables describing the system (population densities, chemical concentrations, etc.)
- λ are parameters (rate constants, birth rates, etc.)

In presence of nonlinearities, no analytic solutions available !

1. Geometric view

Phase space : embedding the evolution of the system into the n-dimensional space spanned by the full set of variables



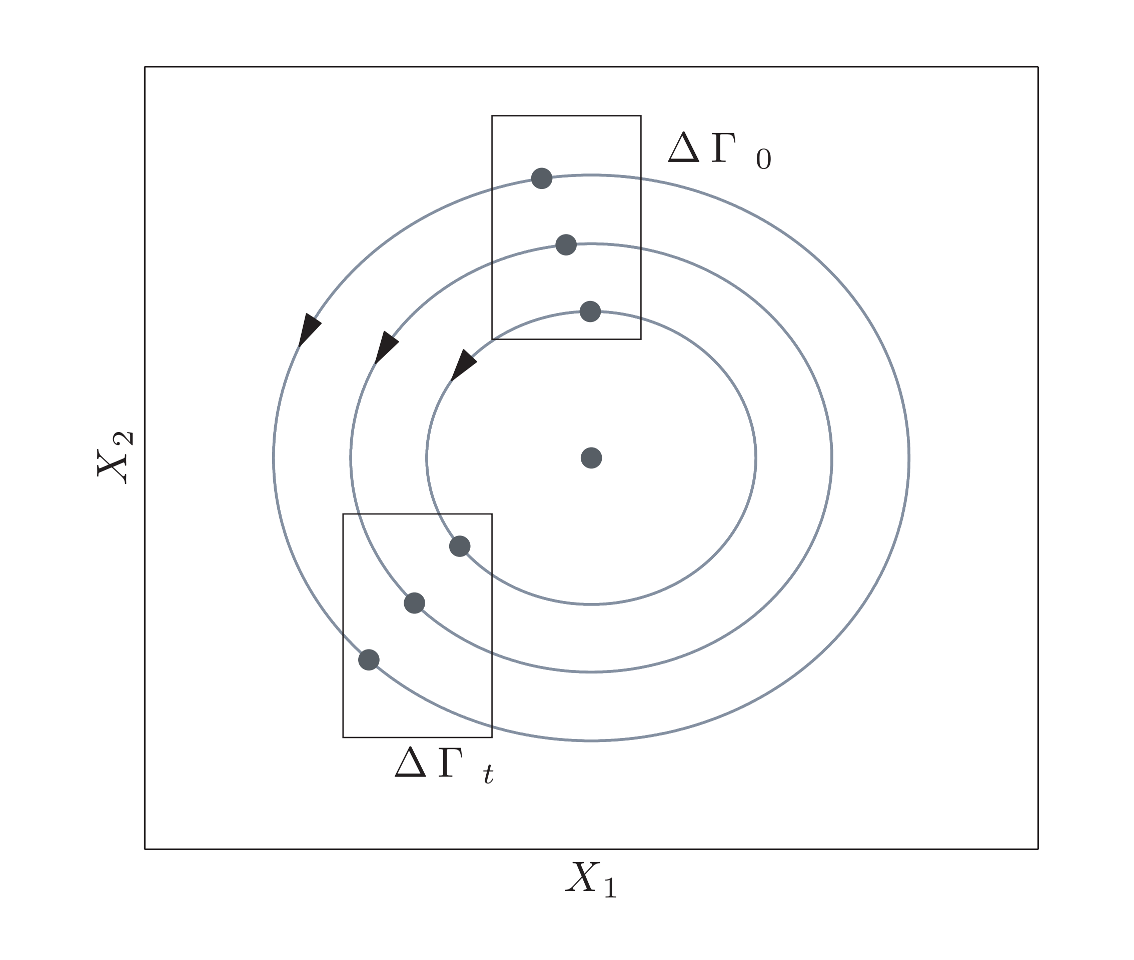
phase space trajectory.
The set of all phase space trajectories will provide all possible behaviors of our system.

Invariant sets
Objects in phase space mapped onto themselves during the time evolution, e.g.

i. **fixed points** (describing the stationary states that can be reached by the system).
ii. **closed curves** (describing a periodic behavior).

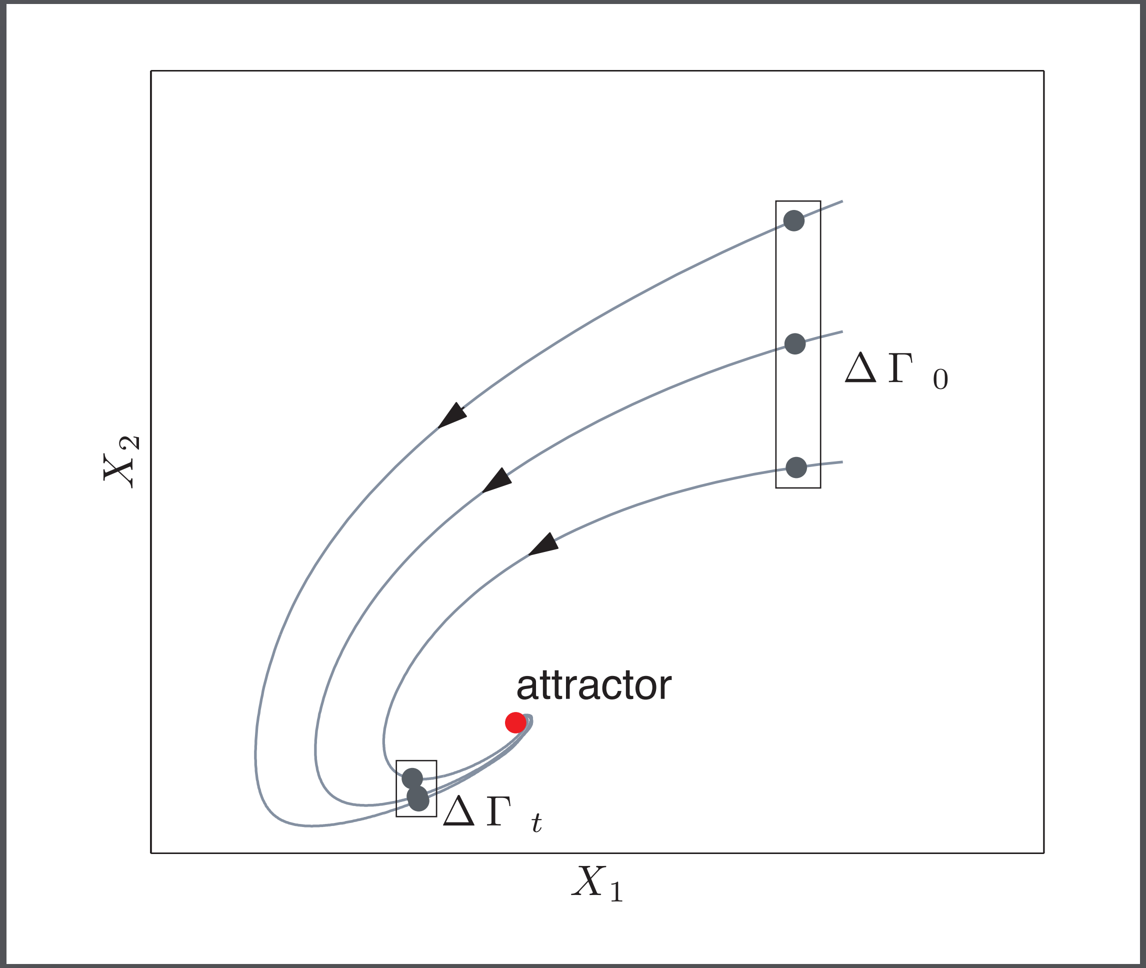
More complex invariant sets include tori and fractals, encountered in systems exhibiting quasi-periodicity or chaos.

Conservative and dissipative systems



Conservative system

$$|\Delta\Gamma_0| = |\Delta\Gamma_t|$$



Dissipative system

$$|\Delta\Gamma_t| < |\Delta\Gamma_0|$$

attractor : ending up on a lower dimensionality

2. Stability

Response to a perturbation removing the system from an initial reference set, *s*

$$X_i(t) = X_{i,s} + \delta x_i(t)$$

- **Stable system :** system remains in a neighborhood of the reference state, s.
- **Asymptotic stability :** $\delta x_i(t) \to 0$ as $t \to \infty$.

Evolution of dynamical systems viewed as a problem of loss of stability of the reference state(s) (e.g., the fixed points) and the emergence of more intricate attractors.

Linear stability analysis

$$\frac{dX_i}{dt} = F_i\left(\{X_j\}, \lambda\right) \qquad j = 1, ...n$$

- Search for reference state, usually among the steady states

$$F_i\left(\{X_{j,s}\}, \lambda\right) = 0$$

- Linearize around $\{X_{j,s}\}$

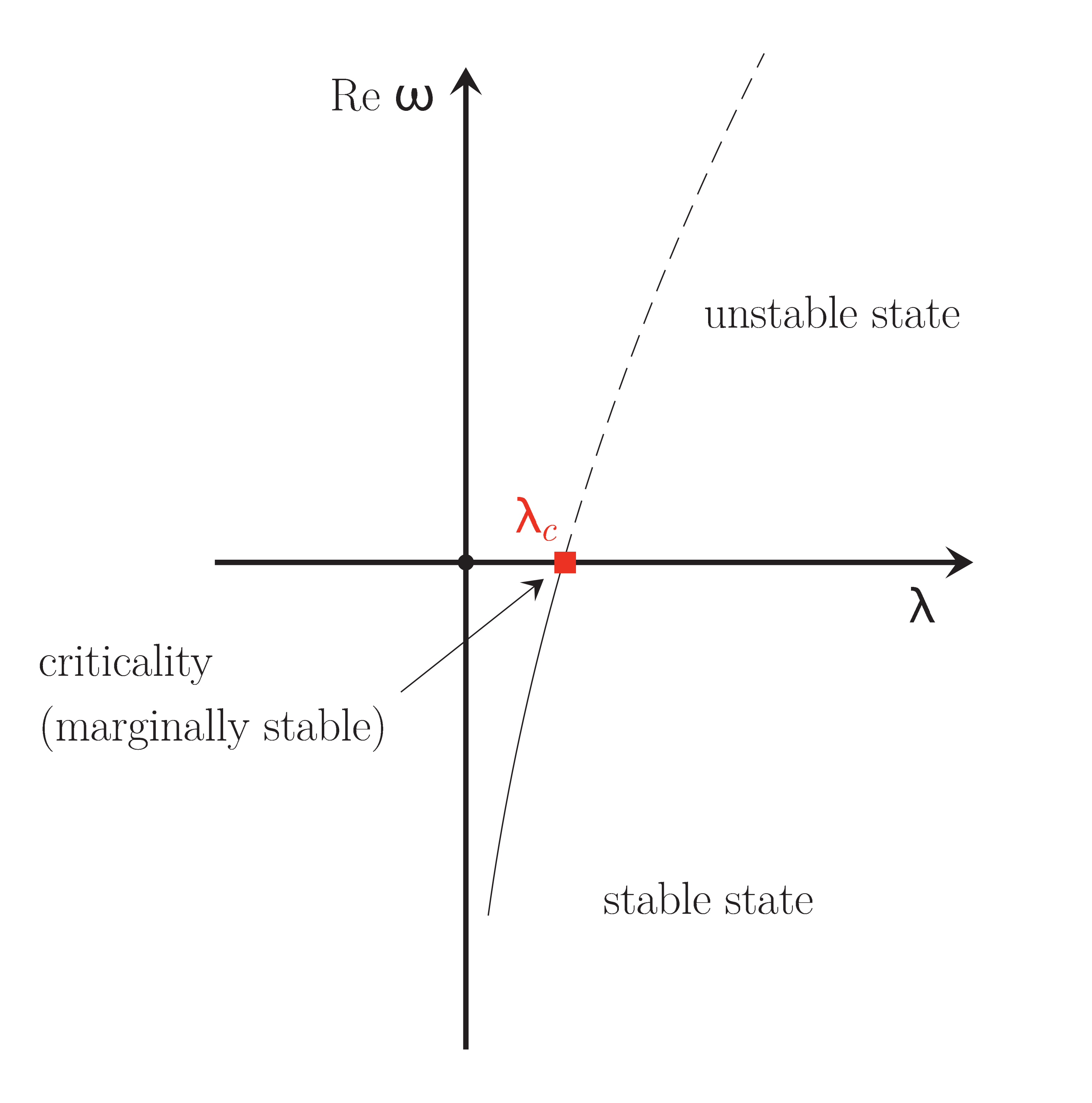
$$X_j = X_{j,s} + \delta x_j \qquad \frac{d\delta x_i}{dt} = \sum_j \left(\frac{\partial F_i}{\partial X_j}\right)_s \delta x_j$$

Solution in the form
$\delta x_i = u_i \mathrm{e}^{\omega_\alpha t}$

- Determine the eigenvalues ω_α ($\alpha = 1, ...n$) of the operator (Jacobian matrix)

$$J_{ij} = \left(\frac{\partial F_i}{\partial X_j}\right)_s$$

as roots of the characteristic equation det $\left|\left(\frac{\partial F_i}{\partial X_j}\right)_s - \omega\delta_{ij}^{\text{kr}}\right| = 0$



Critical value of the parameter at which the eigenvalue changes sign, i.e., the reference state changes stability.

3. Example : Population biology

$$\frac{dX}{dt} = \text{birth} - \underbrace{\text{death}}_{k_2 X} \qquad X \;:\quad \text{population density}$$

1. **Malthus view (18th century) :** birth rate, $k_1 X$

   But, this leads to **explosion** if $k_1 > k_2$
   or to **extinction** if $k_1 < k_2$

2. **Verhulst view (19th century) :** regulated birth rate, $(k_1 - bX)X$

$$\begin{aligned} \frac{dX}{dt} &= k_1 X - bX^2 - k_2 X \\ &= \dots \\ &= kX\left(1 - \frac{X}{N}\right) \end{aligned}$$

where $k = k_1 - k_2$ and $N = k/b$

stationary state :

$$X_{s_1} = 0$$

$$X_{s_2} = N = \frac{k}{b}$$

Stability :

$$X = X_s + \delta x$$

$$\frac{d\delta x}{dt} = \underbrace{\left(k - \frac{2kX_s}{N}\right)}_{(\partial F/\partial X)_s \equiv \omega} \delta x$$

$$\Rightarrow X_{s_1} = 0 \rightarrow \omega = k$$

$$X_{s_2} = N \rightarrow \omega = -k$$

bifurcation diagram :