Lecture 4:

One variable systems

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General features

\[ \frac{dX}{dt} = F(X, \lambda) \]

- 1d phase space → fixed points are only possible attractors
- Real roots \( \omega \) of characteristic equations → monotonic approach towards attractors

\[ \omega = \left( \frac{\partial F}{\partial X} \right)_s \]

\( \omega < 0, \ X_s \) stable

\( \omega > 0, \ X_s \) unstable

\( \omega = 0, \ X_s \) marginally stable (semi-stable)
Complexity is here manifested by the coexistence of more than one simultaneously accessible (i.e., stable) steady states (≡ fixed points).

Transition from single to multiple steady-states? Relative stability?
Canonical example from chemical kinetics
Intuitive approach

3d order autocatalysis in an open well-stirred reactor

\[ A + 2X \xrightarrow{k} 3X \quad (A \text{ in excess}) \quad \Rightarrow \quad \frac{dX}{dt} = kAX^2 - k'X^3 + \frac{1}{\tau}(X_0 - X) \]

Intuitive approach

\[ \frac{dX}{dt} = \underbrace{V(X)}_{\text{production}} + \frac{1}{\tau}(X_0 - X) \quad \text{where} \quad V(X) = kAX^2 - k'X^2 \]
Canonical example from chemical kinetics
Intuitive approach

**Steady states**

\[
kAX_s^2 - k'X_s^3 = \frac{1}{\tau} (X_s - X_0)
\]

- **l.h.s**
- **r.h.s**

- Transition between mono and bi stability: r.h.s. tangent to l.h.s.
Canonical example from chemical kinetics

Analytic view

\[
\frac{dX}{dt} = kAX^2 - k'X^3 + \frac{1}{\tau}(X_0 - X)
\]

4 parameters (too much!). Reduction to two parameters through scaling of \(X\) and \(t\).

\[
x = \frac{X}{X_0} \quad T = tk'X_0^2 \quad \lambda = \frac{kA}{k'X_0} \quad \mu = \frac{1}{\tau k'X_0^2}
\]

\[
\Rightarrow \frac{dx}{dT} = -x^3 + \lambda x^2 - \mu x + \mu
\]
Bifurcation analysis

\[ \frac{dx}{dT} = -x^3 + \lambda x^2 - \mu x + \mu \quad \text{(after scaling)} \]

- Elimination of \( x^2 \) term through transformation \( z = x - \frac{\lambda}{3} \)

\[ \Rightarrow \frac{dz}{dT} = -\left(z + \frac{\lambda}{3}\right)^3 + \lambda \left(z + \frac{\lambda}{3}\right)^2 - \mu \left(z + \frac{\lambda}{3}\right) + \mu \]

or,

\[ \Rightarrow \frac{dz}{dT} = -z^3 + \left(\frac{\lambda^2}{3} - \mu\right) z + \left(\frac{2\lambda^3}{27} - \frac{\mu\lambda}{3} + \mu\right) \quad \text{(I)} \]
Canonical example from chemical kinetics

Bifurcation analysis

First consider case where constant term vanishes. Condition on $\mu$ and $\lambda$ for this

$$\mu = \frac{2\lambda^3}{9(\lambda - 3)} \quad (\lambda > 3, \text{ since } \mu > 0 \text{ for physical reasons})$$

Eq. for $z$ becomes

$$\frac{dz}{dT} = -z^3 + \frac{\lambda^3 - 9\lambda^2}{9(\lambda - 3)} z \quad (\text{II})$$

Steady states:

- $z = 0 \quad (\lambda > 9)$
- $z \pm = \frac{\lambda}{3} \sqrt{\frac{\lambda - 9}{\lambda - 3}} \quad (\lambda > 9)$
- $z = 0 \quad (\lambda < 9)$

pitchfork bifurcation

Notice that trivial state $z = 0$ becomes unstable beyond the bifurcation point $\lambda_c$. The stability of bifurcating branches can be checked straightforwardly (supercritical bifurcation).

This example is in fact paradigmatic: any system in the vicinity of a pitchfork bifurcation can be reduced to eq. (II) (normal form) where $z$ is a combination of the variables (order parameter). All other variables follow $z$ passively.
In the more general case where the constant term in (I) does not vanish, write equation as

$$\frac{dz}{dT} = -z^3 + \lambda z + \mu$$

According to the theory of cubic equations, we have the following situation for the steady states:

![Diagram showing bifurcation analysis for a cubic equation](image)
Canonical example from chemical kinetics
Bifurcation analysis

Limit point bifurcations!

\[\lambda \text{ fixed, (1)}\]

\[\mu \text{ fixed, (2)}\]
Canonical example from chemical kinetics

Kinetic potential and catastrophe theory

A system described by a single variable derives necessarily from a potential, in the sense

\[
\frac{dz}{dT} = -\frac{\partial U}{\partial z}
\]

For our canonical model, \( U \) is obtained by simple quadrature:

\[
U = \frac{z^4}{4} - \frac{\lambda z^2}{2} - \mu z
\]

Correspondence to stability:

- \( z_s \) stable, \( U \) min
- \( z_s \) unstable, \( U \) max
Canonical example from chemical kinetics  
Kinetic potential and catastrophe theory

Transition from one to two stable steady-states

\[ \mu < \mu_c \quad \rightarrow \quad \mu = \mu_c \quad \rightarrow \quad \mu > \mu_c \]

Relative stability:
basins of attraction of the two stable states, or depth of the minimum of the potential.

The concept of structural stability:
classify qualitatively different behaviors that remain robust upon slight changes of the control parameters by determining how the potential is deformed when these parameters are changing.
Catastrophes:

Situations separating qualitatively different behaviors (e.g., cusp point, middle curve of previous slide)

- Full classification possible for cubic nonlinearities as long as two control parameters are available.
- More involved situations for higher order nonlinearities or for multi-variate systems: catastrophe theory.
Population dynamics

Verhulst equation

\[
\frac{dX}{dt} = kX \left(1 - \frac{X}{N}\right)
\]

\( F(X) \)

Fixed points:

\[
X_{s_1} = 0
\]

\[
X_{s_2} = N
\]

Stability:

\[
X = X_s + x
\]

\[
\frac{dx}{dt} = \left(k - \frac{2kX_s}{N}\right)x
\]

\( (\partial F/\partial X)_s \equiv \omega \)
Population dynamics

- $X_{s_1} = 0 \Rightarrow \omega = k$
- $X_{s_2} = N \Rightarrow \omega = -k$

Exchange of stability:

Analytic solution for $k > 0$:

$$X(t) = N \frac{X(0)}{X(0) + (N - X(0))e^{-kt}}$$

Transcritical bifurcation at criticality $k = 0$. 

Bifurcation diagram

Stable

Unstable

$k$
Population dynamics
Comparison with data on population growth

Human population:

\[ N(t) = \frac{246.5 \times 10^6}{1 + 2.243 \exp \left[ -0.02984 (t - 1900) \right]} \]

Figure 5 Population of U.S. Logistic curve fitted so that observed points at 1840, 1900 and 1960 are exact. Points represent census data.

Material production:

Figure 10 Logistic growth of raw material production, showing oscillation on attaining ceiling conditions (Data from S. G. Lasky, Eng. Mining J., 156 (Sept., 1955).)
Global energy balance and climatic change

Variability of earth’s climate over geological time scale. Quaternary glaciations interrupted by interglacial periods.

Does earth’s climate admit multiple states?

Energy balance equation on global scale :

\[ C \frac{dT}{dt} = Q [1 - a(T)] - \epsilon \sigma T^4 \]

where

- \( C \) : heat capacity
- \( G(T) \) : incoming minus reflected
- \( a(T) \) : reflectivity (albedo)
- \( \epsilon \) CO\(_2\) effect
- \( \sigma T^4 \) : Stefan Boltzmann law
Global energy balance and climatic change

Expected form of $a$ : (ice-albedo feedback)

Graphic representation of steady state solutions :

⇒ limit point bifurcation