The sporadic group of Suzuki and apartments in coset geometries

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Chapter 1

Introduction

We first expose the main goals of the present dissertation. Then we introduce the famous Classification of the Finite Simple Groups that plays a kind of starting point for this text. Next we see that Tits’ Building Theory gives a geometric interpretation of almost all the finite simple groups, the exceptions being the alternating groups and the 26 sporadic simple groups. Then we present the project launched by Buekenhout in the eighties that aims to give a unified geometric interpretation of all finite simple groups. Finally we present the structure of the dissertation.

1.1 The main goals

This master’s thesis has two purposes:

- to study a concept of apartment in coset geometries derived from Building Theory; and
- to investigate the geometry of the sporadic simple group of Suzuki.

The first aim is to contribute to a program launched by Buekenhout in the eighties whose final goal is to give a unified geometric interpretation of all the finite simple groups. Namely we study the property (Apt) in coset geometries. The property (Apt) is introduced in Section 2.10. We develop this study under two different angles: from a theoretical point of view with the help of groups and from an algorithmic approach.

The second goal is to investigate the geometry of the sporadic simple group of Suzuki through examples. For this purpose we present some
geometries collected in various sources (see Chapter 5). We also study the property \( (\text{Apt}) \) for those geometries.

1.2 The main results

This master’s thesis contains original mathematical and algorithmic results due to Buekenhout, Leemans and the author.

First we develop and provide an implementation of an algorithm that determines the gonality of a coset geometry. Thanks to it, we were able to find two diagrams for rank 2 geometries for the group \( M_{24} \) that were unknown until now.

Secondly we develop new algorithms to find apartments in coset geometries. The algorithmic developments relative to \( (\text{Apt}) \) are made possible thanks to a new bottom-up approach; it is detailed in Chapter 4. The resulting improvements are considerable. These accomplishments are the subject of a preprint submitted for publication [24].

Thirdly we present a complete list of \( BCDL \)-geometries that satisfy the axiom \( (\text{Apt}) \) for the nine smallest sporadic groups.

Finally we determine for five geometries for the group of Suzuki whether or not they satisfy our concept of apartment.

1.3 The Classification of the Finite Simple Groups

One of the most spectacular mathematical results of all times is the famous Classification of the Finite Simple Groups (CFSG). This project is based on the idea that the finite simple groups form the atomic structure of the finite groups in a similar spirit than the role played by prime numbers in Number Theory. This idea is stated more mathematically by the Jordan-Hölder Theorem. The formulation is the one that appears in Aschbacher [2].

**Theorem 1.3.1.** \( (\text{Jordan-Hölder}) \) All composition series for a group \( G \) have the same length and the same (unordered) family of simple factors.

In 1972, Gorenstein [30] launched a program consisting in 16 steps that would lead to a complete classification of the finite simple groups.
In 1983, he announced that the CFSG was complete. This huge theorem holds in 15,000 pages. It makes the object of a series of books published by the American Mathematical Society (AMS) and whose main authors are Gorenstein, Lyons and Solomon. So far, six volumes have been published. However it happens that Gorenstein’s claim was somewhat premature. Indeed a gap was found and took years of work to fill in thanks to results on quasithin groups due mainly to Aschbacher and Smith (see Aschbacher [2]). The CFSG is certainly one of the most complicated results in mathematics though questioned by some mathematicians for two main reasons. The first reason is the size of its proof: it makes it very difficult to read. The second reason is that some parts depend on computer arguments. However many specialists believe that it is indeed complete and use it [2]. This is our point of view in this thesis too. For the record, we now state the CFSG as it appears in Wilson [61].

**Theorem 1.3.2.** (CFSG) Every finite simple group is isomorphic to one of the following:

1. a cyclic group $C_p$ of prime order $p$;

2. an alternating group $A_n$, for $n \geq 5$;

3. a classical group:
   - linear: $\text{PSL}_n(q), n \geq 2$, except $\text{PSL}_2(2)$ and $\text{PSL}_2(3)$;
   - unitary: $\text{PSU}_n(q), n \geq 3$ except $\text{PSU}_3(2)$;
   - symplectic: $\text{PSp}_{2n}(q), n \geq 2$, except $\text{PSp}_4(2)$;
   - orthogonal: $\text{PO}_{2n+1}(q), n \geq 3$, $q$ odd; or $\text{PO}^+_n(q), n \geq 4$; or $\text{PO}_{2n}(q), n \geq 4$

  where $q$ is a power $p^a$ of a prime $p$;

4. an exceptional group of Lie type:

   $G_2(q), q \geq 3$; $F_4(q)$; $E_6(q)$; $^2E_6(q)$; $^3D_4(q)$; $E_7(q)$; $E_8(q)$

   where $q$ is a prime power, or

   $^2B_2(2^{2n+1}), n \geq 1$; $^2G_2(3^{2n+1}), n \geq 1$; $^2F_4(2^{2n+1}), n \geq 1$

   or the Tits group $^2F_4(2)'$;
5. one of the 26 sporadic simple groups:

- the five Mathieu groups $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$, $M_{24}$;
- the seven Leech lattice groups $Co_1$, $Co_2$, $Co_3$, $McL$, $HS$, $Suz$, $J_2$;
- the three Fischer groups $Fi_{22}$, $Fi_{23}$, $Fi'_{24}$;
- the five monstrous groups $M$, $B$, $Th$, $HN$, $He$;
- the six pariahs $J_1$, $J_3$, $J_4$, $O'N$, $Ly$, $Ru$.

Conversely, every group in this list is simple, and the only repetitions are

\[
\text{PSL}_2(4) \cong \text{PSL}_2(5) \cong A_5; \\
\text{PSL}_2(7) \cong \text{PSL}_3(2); \\
\text{PSL}_2(9) \cong A_6; \\
\text{PSL}_4(2) \cong A_8; \\
\text{PSU}_4(2) \cong \text{PSp}_4(3).
\]

Wilson [61] published recently a book that aims to explain the statement of the CFSG.

A second and a third revision of the CFSG have started. The goal is to produce new approaches to large parts of the proof.

1.4 Sporadic groups

The CFSG gives a list of infinite families of finite simple groups and 26 groups that are called sporadic. However, the definition of a sporadic group to be an item in a list is not really satisfactory. Let us quote Solomon [52]:

I would like to return in 100 years and ask: ‘What do the sporadic simple groups really mean?’ As Hilbert would say: we must know! We will know!

We believe that Incidence Geometry can provide a useful tool to answer at least partially to Solomon’s question. In particular Diagram Geometry already led to promising results [9, 10, 11, 47]. One of the goals of our dissertation is to contribute to this theory by studying the introduction of the axiom (APT).
1.5 The long run goal

Group Theory and Incidence Geometry are related in many ways. However an important problem in Mathematics is to find, in the words of Solomon [52],

"an elegant set of axioms, in the spirit of Tits’ axioms for a building, defining a class of geometries for all the finite simple groups and perhaps more, but not too much more."

Many mathematicians have worked on such a program so far. Despite it, the goal seems still very distant. One of the major difficulties is provided by the sporadic groups. In 1979, Buekenhout [7] invented his diagrams and tried to apply them in a systematic way to a large variety of groups, including the sporadic groups. This resulted in a huge collection of geometries [9] as well as many more other works due to a great number of authors (see Buekenhout and Pasini [18]).

At the Université Libre de Bruxelles (ULB), a group led by Buekenhout has worked on this program for several years. The team has changed slightly over the years. It has been mainly composed of Buekenhout, Cara, De Saedeleer, Dehon, Leemans for the past few years. Nowadays it is composed of Buekenhout, Connor, De Saedeleer and Leemans.

Buekenhout and his team took the groups of Lie-Chevalley type and the buildings as a model. They have slowly elaborated a list of axioms to require in order to select interesting geometries for each group, but not too many. A basic set of three axioms was fixed early and has been used for many years (see Section 1.7). However this basic set is not enough: by requiring so few axioms there is no hope to classify the wild classes of geometries that arise. Therefore Buekenhout and his team have been looking for more axioms. We refer the reader to Buekenhout, et al. [13] for a complete discussion of the work of this team up to 2003. The set of axioms is still not complete and under investigation. This dissertation is a contribution to its development.

1.6 Computational algebra as a tool for experimental research

We now follow Leemans [44].
“In the 1980’s, Buekenhout decided to start a systematic search of geometries constructed from some small groups using an algorithm due to Jacques Tits. With help of Valérie Gobbe, Michel Hermand and Michel Dehon, he started experimentation with small groups using the computational algebra software CAYLEY [21] in order to find all the geometries of rank greater or equal to 3 satisfying some conditions. It showed that their number tends to grow rather wildly but it nevertheless led to algorithms and to a possible, though modest, implementation in CAYLEY. This implementation has been improved over the years. Dehon and Leemans [28, 44] give descriptions of some of the latest and most efficient programs we use nowadays to classify geometries for a given group. Buekenhout, Cara and Dehon used these programs to build an atlas of primitive geometries for small almost simple groups [12].”

“In the same spirit, Buekenhout, Dehon and Leemans [15] built an atlas of residually weakly primitive geometries for some small primitive groups. Later on, Leemans gave another atlas of residually weakly primitive and locally two-transitive geometries for sporadic groups in [44].”

Computational algebra is very present in our dissertation. For instance, Chapter 3 and Chapter 4 concentrate on algorithmic developments that led Leemans and the author to the redaction of a preprint submitted for publication [24].

1.7 Axioms

Thanks to the Tits’ algorithm (see Section 2.8) there is a standard way to build a geometry from a group. Given such a geometry $\Gamma$ constructed from a group $G$ and some of its subgroups $(G_i)_{i \in I}$ with $I := \{1, \ldots, n\}$, we require the following axioms.

The geometry $\Gamma$ is

- flag-transitive (FT);
- firm (F);
- residually connected (RC);
- residually weakly primitive (RWPri);
• (IP), i.e. $\Gamma$ has the intersection property in rank 2 residues;

• locally 2-transitive $(2T)_1$.

The axioms $(F)$, $(FT)$ and $(RC)$ have been the basis of the axiomatic for many years. We refer to Chapter 2 for a detailed explanation of each of them.

In Leemans [44] a geometry satisfying the above axioms is called a $BCDL_{2003}$-geometry. In our thesis we simplify this notation: we call such a geometry a $BCDL$-geometry.

Recently, Buekenhout and Leemans [17] aimed at generalizing the concept of apartment in buildings to incidence geometries. One of the goals of this thesis is to study whether or not it is reasonable to add the following requirement:

• $\Gamma$ satisfies $(Apt)$

to the list of axioms (see Section 2.10.5). Other possible axioms have been studied; we refer for instance to the study of the axioms $(SN)$ and $(BSN)$ in De Saedeleer [26] (see Section 2.9.9).

1.8 Structure of the dissertation

This master’s thesis is divided in five parts. We present each of them by emphasizing the main ideas and results. Before introducing in depth each chapter, we provide a brief summary of each of them.

Chapter 2 is a synthesis of the basic concepts and the fundamental ideas of Incidence Geometry. The results that we present are taken from various sources. However some of them are endowed with a proof due to the author. Chapter 3 presents an algorithm that determines the gonality of a coset geometry. Chapter 4 contains original results due to Leemans and the author: a new algorithm to find apartments in coset geometries. Chapter 5 deals with a detailed presentation of five geometries for the group $Suz$. We provide material for their constructions and their study. Chapter 6 is a presentation of the algorithms used for our research. They are implemented in MAGMA [3].

Chapter 2

We provide definitions of all the concepts and objects that we need in later parts. We try to show to the reader the interest of Incidence
Geometry and the importance that such a theory can have in Mathematics. We introduce the notions of a geometry and a coset geometry. We emphasize the close link that does exist between the geometric point of view and the point of view of groups in Incidence Geometry. We also insist on the notion of the diagram associated to a geometry: it provides a nice way to encapsulate a lot of information relative to the geometry. This encapsulation requires the concept of a residue that forms the atomic structure to build geometries.

In order to refine those concepts, we define (and illustrate through examples) some properties that a geometry can possess. From these properties, we state theorems that characterize classes of geometries.

Chapter 3

Chapter 3 presents a new algorithm that determines the gonality of a rank 2 coset geometry. The mathematical developments presented in this chapter can be adapted to also calculate the diameters of the geometry. Such an algorithm permitted to answer two open questions from Leemans [44]: we determined the gonality and the diameters of two diagrams, incomplete so far.

Chapter 4

Chapter 4 is relative to computational algebra. In order to check the property (APT) with a computer, we develop new algorithms. Such an algorithm already exists in [17] though it is very greedy both in time and memory. We present a bottom-up approach that proves itself much faster as shows Table 5.3.5. This new approach has been very fruitful: we obtained results that were open questions after Buekenhout and Leemans [17]. Chapter 4 makes the object of a preprint that Dimitri Leemans and the author submitted recently for publication [24].

Chapter 5

In Chapter 5, we focus on the sporadic simple group of Suzuki (denoted by $Suz$). We introduce several geometries for this group. We pay special attention to five of them: a geometric construction (or a sketch of the construction) is sometimes given, though a clear construction through cosets is always provided. Also we confront those geometries to the list of axioms we mentioned earlier. Through proofs and developments, we
1.9 Acknowledgements

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Chapter 2
Definitions and fundamental results

In this chapter, we define the concepts we shall need later in this thesis. We illustrate them with examples and we try to show how natural and useful most of them are. Also, we give many fundamental results that help the reader to understand the role of those concepts in Incidence Geometry and that give necessary or sufficient conditions to check them. Most of those results are completed with a proof.

We often use the concept of a graph without defining it. We assume that the reader is familiar with the notions of a graph, a vertex, an edge, a path in a graph, and so on. The reference we chose for the basic terms of Graph Theory that are not defined here is Diestel [29].

2.1 Group theoretical notations

We follow the usual conventions of notations in Group Theory as they are given for instance in the Atlas [25]. However, it is suitable to fix the notations that we use here anyway in order to avoid ambiguity.

When writing $G = A.B$, we mean that $A$ is a normal subgroup of $G$ (which we write $A \triangleleft G$) and that $G/A \cong B$. We use $A \times B$ to denote the direct product of the two groups $A$ and $B$ and $A : B$ denotes a semi-direct product. Whenever we write $G = A:B$, it means that $G = A.B$ and that $G$ is not a semidirect product of $A$ and $B$.

A cyclic group of order $p$ is denoted by $C_p$ or simply by $p$ when no confusion is possible. The notation $p^n$ stands for the elementary abelian
group, i.e. a direct product of cyclic groups $C_p$. The notation $p^{a+b}$ means $p^a.p^b$.

We denote by $D_{2n}$ the dihedral group of order $2n$. We also denote with $D_4$ the group known under the name Klein Vierergruppe; we see this group as the symmetry group of a digon. We denote the symmetric group of order $n!$ by $S_n$ and the alternating group of order $\frac{n!}{2}$ by $A_n$. A Sylow $p$-subgroup of a group $G$ is denoted by $\text{Syl}_p(G)$.

The sporadic simple groups are denoted as in the statement of the CFSG (Theorem 1.3.2). We also adopt the following usual conventions:

- $U_n(q)$ stands for $\text{PSU}(n,q)$;
- $L_n(q)$ stands for $\text{PSL}(n,q)$.

## 2.2 Incidence systems and geometries

We start with the definition of a pregeometry, also called an incidence system or an incidence structure depending on the sources. The following definitions are taken from Buekenhout [11], Buekenhout and Cohen [14] and Pasini [47].

**Definition 2.2.1.** (Pregeometry) Let $\Gamma$ be a four-tuple $(X, *, t, I)$ where

- $X$ is a set whose elements are called the elements of $\Gamma$;
- $I$ is a set whose elements are called the types of $\Gamma$;
- $t$ is a map from $X$ to $I$ called the type function;
- $*$ is a symmetric and reflexive relation on $X \times X$, called the incidence relation of $\Gamma$.

In such a four-tuple $\Gamma = (X, *, t, I)$, $X$ is the disjoint union of all the $X_i = t^{-1}(i)$, with $i \in I$. The four-tuple $\Gamma$ is called a pregeometry or an incidence system over $I$ if the following condition is satisfied: the restriction of $t$ to any flag of $\Gamma$ is an injection. This amounts to require that any two elements of the same type cannot be incident.

If $A \subseteq X$, we say that $A$ is of type $t(A)$ and of rank equal to the cardinality of $t(A)$. The cotype of $A$ is defined to be $I - t(A)$. The cardinality of $I$ is called the rank of $\Gamma$. The corank of $A$ is the cardinality of $I - t(A)$. 

2.2 Incidence systems and geometries

We call flag of $\Gamma$ any set of elements of $\Gamma$ that are pairwise incident. A flag of cotype $i \in I$ is called a panel. A chamber of $\Gamma$ is a flag of type $I$.

We write $x \ast y$ to indicate that $x, y \in \Gamma$ are incident. We define naturally the incidence graph of the pregeometry $\Gamma$ to be the graph whose vertices are the elements of $X$; given to vertices $v_1$ and $v_2$, we define $\{v_1, v_2\}$ to be an edge whenever $v_1$ and $v_2$ are incident in $\Gamma$. This graph is denoted by $(X, \ast)$.

The concept of an incidence structure englobes a lot of mathematical structures. But such a general concept is actually too wild: practice shows that there is no hope to classify incidence structures without more requirements. Therefore we want something a bit less general, but we want to do this in a very natural way in order to keep the feeling that incidence structures provide. This leads to the definition of a geometry namely a pregeometry with more regularity conditions.

**Definition 2.2.2.** *(Geometry)* A geometry over $I$ is a pregeometry $\Gamma$ over $I$ in which any maximal flag is a chamber. Remark that it is equivalent that any nonmaximal flag can be extended in at least one chamber. A geometry is called firm, and we write $(F)$ for this, (resp. thin, thick) provided that any flag of corank 1 is contained in at least two (resp. exactly two, at least three) distinct chambers of $\Gamma$.

The following definition is inspired from Pasini [47].

**Definition 2.2.3.** *(Chamber system)* Let $\Gamma$ be a geometry. The chamber system $C(\Gamma)$ of $\Gamma$ is the graph having the chambers of $\Gamma$ as vertices together with additional structure. Given two distinct chambers $C$ and $D$, the corresponding vertices are joined by an edge whenever $C \cap D$ is a panel. The additional structure is a coloration of the edges: given two distinct adjacent chambers $C$ and $D$, the two elements of $(C \cup D) - (C \cap D)$ belong to the same type, say $i$. We write $C \sim_i D$ to mean this and we say that $C$ and $D$ are $i$-adjacent. We also state $C \sim_i C$ by convention. It is clear that $\sim_i$ is an equivalence relation. The system of equivalence relations $(\sim_i)_{i=1}^n$ may be viewed as a coloured graph obtained by giving to each edge of the graph $C(\Gamma)$ its appropriate colour $i = 1, \ldots, n$.

A path $\gamma = (C_0, C_1, \ldots, C_m)$ in $C(\Gamma)$ from a chamber $C = C_0$ to a chamber $D = C_m$ is called a gallery of length $m$ from $C$ to $D$.

It is proved in Pasini [47] (Theorem 1.9) that $C(\Gamma)$ is connected.
Example 2.2.4. Let $\Gamma$ be the rank 3 geometry of the cube over $I := \{\text{vertex, edge, face}\}$. There are 8 vertices, 12 edges and 6 faces. The incidence relation is induced by the symmetrized inclusion.

A flag is the empty flag or any singleton, pair or triple of pairwise incident elements of any types. A chamber is a triple constituted of a vertex, an edge and a face which are pairwise incident. It can be easily seen that any flag of corank 1 is contained in exactly 2 chambers. Therefore $\Gamma$ is thin.

One can see the chamber system $\mathcal{C}(\Gamma)$ as the truncated cuboctahedron where vertices are the chambers of $\Gamma$ (see Figure 2.1). We identify three different types of edges. They are the square-hexagon type, the square-octagon type and the hexagon-octagon type. Every vertex is incident to exactly one edge of each type. This is equivalent to say that $\Gamma$ is thin.

In order to see the incidence graph of $\Gamma$, take a truncated cuboctahedron, draw a vertex in the middle of each face and join any two whenever the corresponding faces are adjacent.

\footnote{Source: http://en.wikipedia.org/wiki/File:Truncatedcuboctahedron.jpg}
2.3 Subgeometries and truncations

We formalize the intuitive definition of a subgeometry. Actually, we define two concepts of a substructure obtained from a given geometry: subgeometries and truncations. The following definitions are taken from Buekenhout, et al. [14].

**Definition 2.3.1. (Subgeometry)** Let $\Gamma = (X, \ast, t, I)$ be a pregeometry and let $\Gamma' = (X', \ast', t', I')$ be a second pregeometry such that $X' \subseteq X$, $\ast'$ is the restriction of $\ast$ to $X' \times X'$, $I' \subseteq I$ and $t'$ is the restriction of $t$ to $X'$. We call $\Gamma$ a subpregeometry of $\Gamma$. If $\Gamma'$ is a geometry, we say that it is a subgeometry of $\Gamma$.

Proposition 2.3.2 states that the concept of a subgeometry is well defined.

**Proposition 2.3.2.** Given a geometry $\Gamma$, any subgeometry $\Sigma$ is a geometry.

**Proof.** Straightforward. \qed

**Definition 2.3.3. (Truncation)** Let $\Gamma = (X, \ast, t, I)$ be a geometry and let $J \subseteq I$. Then the $J$-truncation of $\Gamma$ is the subgeometry induced by $t^{-1}(J)$ together with the induced incidence relation and the type function. It is often denoted by $^J\Gamma$.

**Example 2.3.4.** Consider again the rank 3 geometry $\Gamma$ of the cube. Choose a face and consider the four edges and the four vertices adjacent to the face. It provides a subgeometry of rank 3 of $\Gamma$: a square completed with its inner face. If now we forget about the face, we get the well known rank 2 geometry of the square which is again a subgeometry of the cube.

Now consider $\Gamma$ and the subset $I' := \{\text{vertex, edge}\}$ of the type set. The $I'$-truncation of $\Gamma$ is a rank 2 geometry that is actually the 1-skeleton of the cube.

2.4 Morphisms

We formalize the concept of a morphism between geometries. The reference is Buekenhout [11].

Let $\Gamma := (X, \ast, t, I)$ and $\Gamma' := (X', \ast', t', I')$ be two geometries. A morphism $\alpha : \Gamma \rightarrow \Gamma'$ is a mapping $\alpha : X \rightarrow X'$ such that
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1. \( t(x) = t(y) \Leftrightarrow t'(\alpha(x)) = t'(\alpha(y)) \);
2. if \( x \ast y \) then \( \alpha(x) \ast' \alpha(y) \); and
3. if \( \alpha(x) \ast' \alpha(y) \) then there exist \( \tilde{x}, \tilde{y} \in X \) such that \( \alpha(\tilde{x}) = \alpha(x) \), \( \alpha(\tilde{y}) = \alpha(y) \) and \( \tilde{x} \ast \tilde{y} \).

An isomorphism \( \alpha \) of \( \Gamma \) onto \( \Gamma' \) is a bijective application \( \alpha : X \to X' \) such that \( t(x) = t(y) \Leftrightarrow t'(\alpha(x)) = t'(\alpha(y)) \) and \( x \ast y \Leftrightarrow \alpha(x) \ast' \alpha(y) \).

An automorphism of \( \Gamma \) is an isomorphism of \( \Gamma \) onto \( \Gamma \). The set of automorphisms of a geometry \( \Gamma \) forms clearly a group and is denoted by \( \text{Aut}(\Gamma) \). We can identify easily a normal subgroup of \( \text{Aut}(\Gamma) \): the subgroup of the type-preserving automorphisms that is defined as the group of all automorphisms that send an element of \( \Gamma \) onto an element of \( \Gamma \) of the same type. This subgroup is denoted by \( \text{Aut}_I(\Gamma) \). Historically, the group \( \text{Aut}(\Gamma) \) was called the group of correlations of \( \Gamma \) and denoted by \( \text{Cor}(\Gamma) \); \( \text{Aut}(\Gamma) \) denoted the group of type-preserving automorphisms of \( \Gamma \).

2.5 Residues

We define here the fundamental concept of a residue in the context of geometries. This concept can be defined more generally in pregeometries but we only consider geometries in the later parts of this dissertation. The references are Buekenhout [11], Buekenhout, et al. [14] and Pasini [47].

**Definition 2.5.1. (Residue)** Let \( \Gamma = (X, \ast, t, I) \) be a geometry and let \( F \) be a flag of \( \Gamma \). The residue of \( F \) in \( \Gamma \) is defined as the four-tuple \( \Gamma_F = (X_F, \ast_F, t_F, I_F) \) where:

- \( X_F \) is the set of elements of \( X - F \) which are incident with every element of \( F \);
- \( \ast_F \) and \( t_F \) are the restrictions of \( \ast \) and \( t \) to \( X_F \times X_F \) and \( X_F \) respectively;
- \( I_F := I - t(F) \).

If \( F = \{x\} \), where \( x \in X \), we write also \( X_x, \ast_x, t_x, I_x \) and \( \Gamma_x \) instead of \( X_F, \ast_F, t_F, I_F \) and \( \Gamma_F \).
2.5 Residues

Proposition 2.5.2 is taken from Buekenhout [14]. It shows that the residue of a flag is well defined. The proof is due to the author.

**Proposition 2.5.2.** For any geometry $\Gamma = (X, *, t, I)$ and for any flag $F$ of $\Gamma$, the following three statements hold:

1. the residue $\Gamma_F$ is a subgeometry of $\Gamma$;
2. a subset $A \subset X_F$ is a flag of $\Gamma_F$ if and only if $F \cup A$ is a flag of $\Gamma$; and
3. if $A$ is a flag of $\Gamma_F$, then $(\Gamma_F)_A = \Gamma_{F \cup A}$.

**Proof of 1.** By contradiction, consider a maximal flag $A$ in $\Gamma_F$ and suppose it is not a chamber in $\Gamma_F$. By point 2, $F \cup A$ is a flag of $\Gamma$ and it is not a chamber. Since $\Gamma$ is a geometry, $F \cup A$ can be extended to a chamber $F \cup A \cup B$. Again by point 2, $A \cup B$ is flag of $\Gamma_F$. Moreover it properly contains $A$, a contradiction.

**Proof of 2.** Let $A \subset X_F$ be a flag of $\Gamma_F$. By definition, the elements of $F \cup A$ are pairwise incident and pairwise of distinct types. Hence $F \cup A$ is a flag of $\Gamma$. This proves the implication from left to right. For the converse, suppose $F \cup A$ is a flag of $\Gamma$. Then $A \subset X_F$ because every element of $A$ is incident with every element of $F$. Moreover, the elements of $A$ are pairwise incident by hypothesis. Hence $A$ is a flag of $\Gamma_F$.

**Proof of 3.** Let $A$ be flag of $\Gamma_F$. By definition, $(\Gamma_F)_A$ is the set

$$\{ x \in X_F | x * a, \ \forall a \in A \}$$

and $\Gamma_{F \cup A}$ is the set

$$\{ x \in X | x * f, \ \forall f \in F \cup A \}.$$  

Clearly $(\Gamma_F)_A \subseteq \Gamma_{F \cup A}$. Now let $x \in \Gamma_{F \cup A}$. Since

$$x * f, \ \forall f \in F \cup A,$$

it follows in particular

$$x * f, \ \forall f \in F,$$

i.e. $x \in X_F$ and $x * a, \ \forall a \in A$. So

$$(\Gamma_F)_A \supseteq \Gamma_{F \cup A}$$

and thus $(\Gamma_F)_A = \Gamma_{F \cup A}$.

$\square$
Example 2.5.3. In the rank 3 geometry \( \Gamma \) of the cube, consider a flag of rank 1 constituted of only one face. Its residue is a square and is actually the usual rank 2 geometry of the square. The residue of an edge is a digon, namely two faces and two vertices that are pairwise incident, and the residue of a vertex is a triangle.

2.6 Fundamental properties

We now introduce two fundamental properties of the axiomatic developed at the Université Libre de Bruxelles. They are natural regularity properties. They have been the basis of the axiomatic from an early stage of the program launched by Buekenhout.

2.6.1 Connectedness

The references for this section are Buekenhout [11], Buekenhout, et al. [14] and Pasini [47].

Definition 2.6.1. (Connectedness, residual connectedness) Let \( \Gamma = (X, *, t, I) \) be a geometry. We say that \( \Gamma \) is connected if the incidence graph \((X, *)\) is connected. We say that \( \Gamma \) is residually connected, and we write (RC) for this, provided that \((X_F, *_F)\) is connected for any flag \(F\) of \(\Gamma\) of corank at least 2.

Let us remark that if \( \Gamma \) is of rank at least 2, then \( \Gamma \) is the residue of the empty flag. Therefore the residual connectedness of \( \Gamma \) implies its connectedness.

Definition 2.6.2. (Strong connectedness) We say that a geometry \( \Gamma = (X, *, t, I) \) is strongly connected if for any two distinct types \( i, j \in I \) and for any flag \( F \) of \( \Gamma \) with no element of type \( i \) nor \( j \), the \( \{i, j\} \)-truncation of \( \Gamma_F \) is nonempty and connected.

Proposition 2.6.3 is due to Buekenhout and Schwartz [19]. It states that strong connectedness and residual connectedness coincide in geometries of finite rank. The following proof is due to the author.

Proposition 2.6.3. Let \( \Gamma = (X, *, t, I) \) be a geometry of finite rank. Then \( \Gamma \) is residually connected if and only if \( \Gamma \) is strongly connected.
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Proof. First suppose $\Gamma$ is strongly connected. Let $F$ be a flag of $\Gamma$ of corank at least 2, and suppose that $F$ has no element of type $i$ nor $j$. Since the $\{i, j\}$-truncation of $\Gamma_F$ is connected for any $i, j$ as above by hypothesis, we deduce that $X_F$ is connected. This is true for any flag $F$, so $\Gamma$ is residually connected.

For the converse, suppose $\Gamma$ is residually connected. If $F$ is a flag of corank 2, $X_F$ is connected and there is nothing to prove. Suppose $F$ is of corank at least 3. Let $i, j \in I$ be two types such that $F$ has no element of type $i$ nor $j$. Consider a path in $\Gamma_F$ between any two elements $a$ and $b$ of type $i$ and $j$ respectively (such a path exists by residual connectedness of $\Gamma$). Suppose this path contains a vertex of type $k \neq i, j$. Let $y$ be the first such element in the path, let $x$ be the element before it which is thus of type, say, $i$. Let $z$ be the element that comes after $y$ which is of type, say, $l$. Now consider the residue $R_y$ in $\Gamma_F$ of the flag $y$. Since $\Gamma$ is residually connected, the residue $R_y$ is connected. Thus we can find a path between $x$ and $z$ in $(\Gamma_F)_y$ that does not use any element of type $k$. By proceeding inductively from now and since $I$ is finite by hypothesis, it is finally possible to find a path between $a$ and $b$ which only uses vertices of types $i$ and $j$. This proves that the $\{i, j\}$-truncation of $\Gamma_F$ is connected. Since this is true for any $i, j$ as above, $\Gamma$ is strongly connected.

Example 2.6.4. Take the rank 3 geometry $\Gamma$ of the cube. Its incidence graph described in example 2.2.4 is connected, hence $\Gamma$ is connected. Taking any residue of corank 2 or 3, one obtains a cube, a square, a triangle or a digon. Those four are connected. Therefore $\Gamma$ is residually connected.

2.6.2 Flag-transitivity

We say that a geometry $\Gamma := (X, *, t, I)$ is flag-transitive and we write (FT) to mean this, provided that the group of type-preserving automorphisms $\text{Aut}_I(\Gamma)$ is transitive on the chambers of $\Gamma$. Note that this implies that it is transitive on the flags of a given type as well. Let us remark that in a flag-transitive geometry, it is possible to speak about the residue of a given type since all the flags of a given type are in the same orbit of $\text{Aut}_I(\Gamma)$.

Although we mention this important property here, we do not insist on this notion because the introduction of groups thanks to coset geometries provides a better ground to develop this notion.
2.7 The Buekenhout diagram of a geometry

It is possible to associate a diagram to a geometry which we suppose here firm, residually connected and flag-transitive. The interest of such a notion is that the diagram of a geometry encapsulates a lot of information about rank 2 residues of the geometry. It is then possible to get much of the structure of the geometry simply by studying its diagram. Let us cite Pasini [47]:

Borrowing an image from chemistry, geometries of rank 2 are like atoms, but I am interested in the molecules that we can build by connecting two or more of those atoms.

The reader may refer to Buekenhout and Cohen [14] or to Pasini [47] for a deep study of this subject.

Definition 2.7.1. (Diagram) Let $\Gamma = (X, *, t, I)$ be a firm, flag-transitive and residually connected geometry. The diagram associated to $\Gamma$ consists in a nonoriented graph of $|I|$ vertices named after the elements of $I$. Two vertices corresponding to the elements $i, j \in I$ are joined by an edge in the graph if the following two conditions hold:

1. there exists a flag $F$ of type $I \setminus \{i, j\}$ in $\Gamma$; and

2. there exists two elements of type $i$ and $j$ in $\Gamma_F$ which are not incident in $\Gamma_F$.

We say that a geometry belongs to its diagram.

2.7.1 Parameters for rank 2 geometries

It is possible to refine the diagram of a geometry $\Gamma$ of rank 2 with some parameters that characterize the residues and the flags of $\Gamma$.

Definition 2.7.2. (Diameters, gonality, orders) Let $\Gamma = (X, *, t, I)$ be firm, connected geometry of rank 2. We say that $x, y \in X$ are at distance $k$ if they are at distance $k$ in the incidence graph $(X, *)$. For $j \in I$, the $j$-diameter $d_j$ of $\Gamma$ is the greatest number occurring as a diameter of $(X, *)$ to some element of type $j$. The diameter of $\Gamma$ is defined as $d := \max\{d_i \mid i \in I\}$. The gonality is the smallest number $g > 0$ such that $(X, *)$ has a circuit of length $2g$. The order of an element $x$ of type $j$ is $s_j := |\Gamma_x| - 1$. 
2.7 The Buekenhout diagram of a geometry

Those informations are summed up in the diagram of Figure 2.2 where $n_k$ is the number of elements of type $k$ with $k = i, j$.

**Definition 2.7.3.** (Buekenhout diagram) The Buekenhout diagram of a geometry is its diagram as in Definition 2.7.1 together with the parameters given in Figure 2.2.

The Coxeter diagram of a geometry $\Gamma$ is obtained by labelling the edge only with $g$ and is denoted by $\text{Cox}(\Gamma)$.

**Proposition 2.7.4.** Let $\Gamma$ be a rank 2 geometry over $I = \{p, l\}$. Then we have $g \leq \min(d_p, d_l)$ and $d \leq 1 + \min(d_p, d_l)$. Moreover, if $d$ is an odd integer or $d = \infty$, then $d = d_p = d_l$.

**Proof.** If $g = \infty$ then $\Gamma$ is a tree, whence $g = d_p = d_l = \infty$. Let $g < \infty$. We first prove the inequality $g \leq \min(d_p, d_l)$. Let $(v_0, v_1, \ldots, v_{2g} = v_0)$ be a circuit of $\Gamma$ of maximal length $2g$. Interchanging the roles of $p$ and $l$ if necessary, we may always assume that $d_p \leq d_l$ and that $v_0 \in P$. Let $g > d_p$, if possible. Then a path $(v_0, v'_1, \ldots, v'_k = v_g)$ can be found, with $k \leq d_p$. This gives us a closed path $(v_0, v_1, \ldots, v_g, v'_{k-1}, \ldots, v'_1, v_0)$ of length less than $2g$, a contradiction.

Let us turn to the inequality $d \leq 1 + \min(d_p, d_l)$. Given a line $x \in l$ and an element $v$ at maximal distance $d_l$ from $x$, let $(x = v_0, v_1, \ldots, v_m)$ be a path of minimal length $m = d_l$ from $x$ to $v$. Then $(v_1, v_2, \ldots, v_m)$ is a path of minimal length from the point $v_1$ to $v$. Therefore $d_p \geq d_l - 1$. Similarly, $d_l \geq d_p - 1$, interchanging the roles of $p$ and $l$. Therefore $d \leq 1 + \min(d_p, d_l)$. In particular, $d = d_p = d_l = \infty$ if one of $d_p$ or $d_l$ is infinite.

Finally let $d$ be an odd integer and let $x, y$ be elements of $\Gamma$ at distance $d$. Since $\Gamma$ is a bipartite graph, elements of $\Gamma$ at odd distance...
belong to distinct types. Therefore, one of \( x, y \) is a point and the other is a line. Hence \( d = d_p = d_l \).

### 2.7.2 Generalization for geometries of rank \( n \geq 3 \)

We generalize the notions introduced in Section 2.7.1 to geometries of rank \( n \geq 3 \). Our references are Buekenhout, et al. [14] and Leemans [44].

**Definition 2.7.5.** *(Diagram)* The diagram \( D \) over a set of types \( I \) consists in an application \( D \) defined over \( \binom{I}{2} = \{ \{i, j\} \subseteq I \mid i \neq j \} \) assigning to any pair \( \{i, j\} \) of distinct elements of \( I \) a class \( D(i, j) = D(j, i) \) of rank 2 geometries over \( \{i, j\} \).

A geometry \( \Gamma \) belongs to the diagram \( D \) over \( I \) if for any pair \( \{i, j\} \) of distinct elements of \( I \) and for any flag \( F \) of \( \Gamma \) such that \( \Gamma_F \) is of type \( \{i, j\} \), we have \( \Gamma_F \in D(i, j) \). In this case, \( \Gamma \) is said to be of type \( D \).

Let \( J \) be a subset of \( I \). A geometry \( \Gamma \) over \( I \) has a \( J \)-order \( (s_j)_{j \in J} \), where \( s_j \) is an integer, if for any \( j \in J \) and for any flag \( F \) of \( \Gamma \) of type \( I - \{j\} \), the size of \( X_F \) is \( s_j + 1 \).

We use the following conventions. In case the labels of an edge are \( d_i = g = d_j = n \), we write only \( n \) above the edge for any \( i, j \in I \). We call a rank 2 geometry with \( d_0 = g = d_1 = n \) a *generalized n-gon* or more generally a *generalized polygon*. In case \( n = 2 \), we do not draw any edge at all; if \( n = 3 \), we draw the edge without any label, and if \( n = 4 \), we draw two parallel edges without any label. If the labels are \( d_i = 3, g = 3 \) and \( d_j = 4 \), we write \( L \) because the corresponding residue is a linear space, i.e. a point-line geometry where two points have at most one line in common and given a line \( l \) and a point \( p \) not incident to \( l \), there exist a line \( l' \) and a point \( p' \) incident to \( l' \) such that \( p \) and \( p' \) are both incident to \( l' \). Moreover if \( s_i = s_j - 1 \) we write \( Af \) instead of \( L \) because the residue is isomorphic to an affine plane. Finally if \( s_i = 1 \), no matter what the value of \( s_j \) is, we write \( c \) instead of \( L \) because the residue is isomorphic to a complete graph. If \( g_{ij} = d_{ij} = d_{ji} = 5 \), and \( s_i = s_j = 1 \), we write \( P \) in place of these three numbers because the corresponding residue is isomorphic to the so-called Petersen graph. A ‘*’ just after one of these later symbols means that we talk about the dual of the geometry corresponding to the symbol.

**Example 2.7.6.** Let us consider again the rank 3 geometry \( \Gamma \) of the cube. It belongs to the diagram
Now consider the $\{\text{vertex, edge}\}$-truncation of the cube. We obtain a geometry belonging to the diagram

$$
\begin{array}{cccc}
0 & 1 & 2 \\
1 & 1 & 1 \\
8 & 12 & 6
\end{array}
$$

**Example 2.7.7.** The well known Petersen graph is a rank 2 geometry with 10 points and 15 lines belonging to the diagram

$$
\begin{array}{cccc}
0 & 6 & 4 & 6 & 1 \\
1 & 2 \\
8 & 12
\end{array}
$$

This geometry can be obtained from the dodecahedron by identifying pairs of opposite points.

### 2.8 Coset geometries and the Tits algorithm

After Tits [57], there is a standard way to define a geometry from a group and a collection of its subgroups.

Let $\Gamma = (X, \ast, t, I)$ be a geometry and let $G$ be a group of automorphisms of $\Gamma$ acting transitively on the chambers. For any flag $F$, the stabilizer $G_F$ in $G$ is called a *parabolic subgroup*. The stabilizer of an element is called a *maximal* parabolic subgroup. The stabilizer of a panel is called a *minimal* parabolic subgroup. The stabilizer of a chamber is called a *Borel subgroup* and is usually denoted by $B$.

Let $F$ be a flag and let $\Gamma_F$ be of rank $j$. Consider the parabolic subgroup $G_F$. By $P(G_F)$ we denote the subgroup of $G_F$ fixing $\Gamma_F$
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elementwise. In the following parts of the dissertation, we emphasize it by placing it between brackets: \( G_i = [P(G_i)] \cdot G_i / P(G_i) \).

Fix a chamber \( F = \{ x_i \mid i \in I, t(x) = i \} \). Let \( P_i = G_{x_i} \). For each element \( y_i \) of type \( i \) in \( \Gamma \), there is some \( g \in G \) with \( g(x_i) = y_i \), and \( y_i \) can be identified with the coset \( P_i g \). Now, let \( y_i \) be identified with \( P_i g \) and let \( z_j \) be identified with \( P_j h \). Then \( y_i \ast z_j \) if and only if \( P_i g \cap P_j h \) is nonempty in \( G \). Moreover, if \( a \in G \), then \( a \) transforms \( y_i = P_i g \) in \( P_i g a \). Hence the geometry \( \Gamma \) is entirely described by the data of \( G \) and of the subgroups \( P_i, i \in I \).

Conversely, given a group \( G \) and a family of subgroups \( (G_i)_{i \in I} \), we can define a pregeometry \( \Gamma = \Gamma(G, \{ G_i \mid i \in I \}) \) over \( I \) as follows. The elements of type \( i \), for \( i \in I \), are the right cosets \( G_i g \). Two elements \( G_i g \) and \( G_j h \) are incident if and only if \( G_i g \cap G_j h \) is nonempty. The group \( G \) acts on \( \Gamma \), by left translation, as a group of automorphisms. The action of \( G \) is transitive on all flags of cardinality 2 of a given type but it needs not be flag-transitive.

We just proved the following construction.

**Proposition 2.8.1.** (Tits [57]) Let \( G \) be a group and let \( (G_i)_{i \in I} \) be a collection of subgroups of \( G \). Let \( X = \{ G_i g \mid i \in I, g \in G \} \), \( t : X \to I : G_i g \mapsto i \) and \( * = \{ (G_i g, G_j h) \mid G_i g \cap G_j h \neq \emptyset \} \). Then \( \Gamma \) is a pregeometry having a chamber. Moreover, \( G \) acts by right multiplication as an automorphism group of \( \Gamma \). The group \( G \) is transitive on the flags of rank lower or equal to 2.

The action of \( G \) is flag-transitive if and only if every family of cosets \( (x_i G_i)_{i \in J}, J \subseteq I \), which have pairwise a nonempty intersection, have some common element.

Now given a group \( G \) and a collection of its subgroups \( \{ G_i \mid i \in I \} \), we denote by \( \Gamma(G, \{ G_i \mid i \in I \}) \) the pregeometry defined above. We then refer to it simply by \( \Gamma \). If \( \Gamma \) is a geometry, we call it a coset geometry.

The Tits algorithm gives a really powerful tool for the study of geometries, in particular their construction if they are flag-transitive. Now Group Theory is available to study geometries and to compute them easily in computers.

We follow the following conventions throughout this thesis. Let \( \Gamma(G, (G_i)_{i \in I}) \) be a coset geometry. Denoting by \( \alpha \) the word \( i_1 \ldots i_k \) with \( \{ i_1, \ldots, i_k \} \subseteq I \), we write \( G_\alpha \) for \( G_{i_1 \ldots i_k} \) (which is in itself an
abbreviation for $G_{\{i_1, \ldots, i_k\}}$ to mean $G_{i_1} \cap \ldots \cap G_{i_k}$. The set of the minimal parabolic subgroups of $\Gamma$ is then denoted by $\{G_{I-i} \mid i \in I\}$ for $\{G_{I-\{i\}} \mid i \in I\}$, for short.

## 2.9 Properties of a coset geometry

In this section, we define some properties relative to coset geometries and we study them. Those properties are the axioms we mentioned in Chapter 1. The motivations for requiring such properties are presented in Buekenhout, et al. [13] and the list is still not complete as we already explained in Chapter 1. The main references are Buekenhout [11], Buekenhout, et al. [14, 12, 13] and Pasini [47].

### 2.9.1 Flag-transitivity

Let $G$ be a group, let $\{G_i \mid i \in I\}$ be a collection of subgroups of $G$ and let $\Gamma(G, \{G_i \mid i \in I\})$ be a coset geometry. We denote by $F$ a flag of type $J$ in $\Gamma$. The following result gives a way to check whether $\Gamma$ is a flag-transitive geometry. See also Dehon [27] and Hermand [34].

**Lemma 2.9.1.** (Buekenhout, Hermand [34]) Let $\mathcal{P}(I)$ be the set of all the subsets of $I$ and let $\alpha: \mathcal{P}(I) - \{\emptyset\} \rightarrow I$ be a function such that $\alpha(J) \in J$ for every $J \subset I$, $J \neq \emptyset$. the geometry $\Gamma$ is flag-transitive if and only if, for every $J \subset I$ such that $|J| \geq 3$, we have

$$
\bigcap_{j \in J-\alpha(J)} (G_j G_{\alpha(J)}) = \left( \bigcap_{j \in J-\alpha(J)} G_j \right) G_{\alpha(J)}
$$

Also in Pasini [47], a necessary and sufficient condition is given to check flag-transitivity if the geometry is connected. It is Lemma 2.9.2. The proof is taken from [47].

**Lemma 2.9.2.** The group $G$ is flag-transitive if and only if, for every $i \in I$, the subgroup $G_{I-i}$ is transitive on the set of chambers containing the panel $F_{I-i}$.

**Proof.** The implication from left to right is trivial. Let us prove the other implication. Let $G_{I-i}$ be transitive on the set of chambers containing $F_{I-i}$, for every $i \in I$. Given a chamber $C$, let $C_0, C_1, \ldots, C_m = C$
be a gallery from $C_0$ to $C$. We prove by induction on $m$ that there is an element $g \in G$ mapping $C_0$ onto $C$. If $m = 0$, there is nothing to prove. Let $m \geq 1$. By inductive hypothesis, there is an element $g_1 \in G$ mapping $C_0$ onto $C_{m-1}$. Let $i$ be the cotype of the panel $C_{m-1} \cap C_m$. The chambers $C_0 = g_1^{-1}(C_{m-1})$ and $g_1^{-1}(C_m)$ are $i$-adjacent. Therefore, there is an element $g_2 \in G_{i-1}$ mapping $C_0$ onto $g_1^{-1}(C_m)$, by assumption. Hence $g_1g_2$ maps $C_0$ onto $C_m$. 

In Tits [59], an ‘easy’ exercise is given in the first chapter. Many mathematicians tried to solve it and realize that this exercise is not as easy as it is stated to be. We give it as Lemma 2.9.3. It is a very useful result strongly related to the concept of flag-transitivity. For instance, Dehon [27] used Lemma 2.9.3 to simplify the calculations occurring after implementing an algorithm based on Lemma 2.9.1. The proof of Lemma 2.9.3 that we give is due to Horn [36].

**Lemma 2.9.3.** (Tits [59]) Let $G_1, G_2, G_3$ be three subgroups of a group $G$. Then the following conditions are equivalent.

1. $G_2G_1 \cap G_3G_1 = (G_2 \cap G_3)G_1$
2. $(G_1 \cap G_2) \cdot (G_1 \cap G_3) = (G_2G_3) \cap G_1$
3. If the three cosets $xG_1$, $yG_2$ and $zG_3$ have pairwise nonempty intersection, then $xG_1 \cap yG_2 \cap zG_3 \neq \emptyset$.

**Proof.** (1) implies (2). Indeed, the left side of (2) is obviously contained in the right side. So pick an arbitrary $h = g_2g_3 = g_1 \in (G_2G_3) \cap G_1$. Now, $g_3 = g_2^{-1}g_1$ is contained in $G_2G_1$ but also in $G_3G_1$ and hence by (1) in $(G_2 \cap G_3)G_1$. So there is $g_23 \in G_2 \cap G_3$ and $g_1 \in G_1$ such that $g_3 = g_2^{-1}g_1 = g_2g_3g_1$. Thus $g_2g_3 = g_1g_1^{-1}g_1 \in G_1 \cap G_2$ and $g_2^{-1}g_3 = g_1 \in G_1 \cap G_3$, therefore $h = g_2g_3 = (g_2g_3)(g_2g_3^{-1}) \in (G_1 \cap G_2) \cdot (G_1 \cap G_3)$.

(2) implies (3). Without loss of generality assume $x = 1$. From $G_1 \cap yG_2 \neq \emptyset$ follows $y \in G_1G_2$; similarly we find $z \in G_1G_3$.

Thus $y = g_1g_2$, $z = g_1g_3$ for suitable elements $g_1 \in G_1$, $g_2 \in G_2$, $g_3 \in G_3$.

From $yG_2 \cap zG_3 \neq \emptyset$ we get $y^{-1}z \in G_2G_3$, hence

$$g_2^{-1}g_1^{-1}g_1g_3 \in G_2G_3 \implies g_1^{-1}g_1 \in (G_2G_3) \cap G_1.$$ 

But using (2), without loss of generality we may assume $g_1 \in G_1 \cap G_2$ and $g_1 \in G_1 \cap G_3$. Thus $y \in G_2$ and $z \in G_3$, and so $xG_1 \cap yG_2 \cap zG_3 = G_1 \cap G_2 \cap G_3 \neq \emptyset$. 

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(3) implies (1). The right side of (1) is clearly contained in the left side, so we just need to prove the reverse inclusion. So assume 
\( h \in G_2G_1 \cap G_3G_1 \). Then \( h \in G_2y \cap G_3z \) for suitable \( y, z \in G_1 \). Thus 
\( y \in G_1 \cap yG_2 \) and \( z \in G_1 \cap zG_3 \). Thus by (3) there exists \( g_1 \in G_1 \cap G_2y \cap G_3z \), and we have 
\( 1 \in G_2yg_1^{-1} \cap G_3zg_1^{-1} \). This implies \( yg_1^{-1} \in G_2 \) and \( zg_1^{-1} \in G_3 \). Therefore 
\( hg_1^{-1} \in G_2yg_1^{-1} \cap G_3zg_1^{-1} = G_2 \cap G_3 \), and thus 
\( h \in (G_2 \cap G_3)G_1 \) as claimed.

Remark that the first point of Lemma 2.9.3 is a particular case of Lemma 2.9.1.

In the software Magma \[3\], Leemans implemented an algorithm to check flag-transitivity based on Lemma 2.9.4 and Theorem 2.9.5. Lemma 2.9.3 is the key to prove them. The proofs are due to Leemans \[40, 41\].

**Lemma 2.9.4.** Let \( \Gamma(G, \{A, B, C\}) \) be a flag-transitive rank 3 coset geometry, let \( H \) be a subgroup of \( G \). Then the pregeometries \( \Gamma_A(G, \{A, B, H\}), \Gamma_B(G, \{A, H, C\}) \) and \( \Gamma_C(G, \{H, B, C\}) \) are flag-transitive geometries if and only if the pregeometry \( \Gamma''(G, \{A, B, C, H\}) \) is a flag-transitive geometry.

**Proof.** The implication from right to left follows from the fact that \( \Gamma_A, \Gamma_B \) and \( \Gamma_C \) are truncations of \( \Gamma'' \) which is supposed to be flag-transitive. For the converse, first let \( \Gamma'(H, \{A \cap H, B \cap H, C \cap H\}) \) and observe that this geometry is flag-transitive. Now in order to see the result, it is enough to check that

\[
AC \cap BC \cap HC \subseteq (A \cap B \cap H)C. \tag{2.9.1}
\]

We now observe that

\[
(A \cap B \cap H)C = ((A \cap H) \cap (B \cap H))(C \cap H)C
\]
\[
= ((A \cap H)(C \cap H) \cap (B \cap H)(C \cap H))C
\]
\[
= (H \cap AC \cap BC)C
\]

where the equalities hold because \( \Gamma', \Gamma_A, \Gamma_B \) and \( \Gamma_C \) are flag-transitive.

So we can rephrase equation 2.9.1 as

\[
AC \cap BC \cap HC \subseteq (H \cap AC \cap BC)C \tag{2.9.2}
\]

Let \( t \in AC \cap BC \cap HC \). Clearly, \( t = hc \) for some \( h \in H, c \in C \). In particular, \( t \in AC \) and \( t \in BC \). Hence \( tc^{-1} = hcc^{-1} = h \). Consequently \( h \in AC \) and \( h \in BC \).

\( \square \)
Theorem 2.9.5. Let $\Gamma(G, \{G_1, \ldots, G_n\})$ be a flag-transitive coset geometry of rank $n$, let $H$ be a subgroup of $G$ and let $\Gamma'(H, \{G_1 \cap H, \ldots, G_n \cap H\})$ be a flag-transitive geometry. Then the pregeometries $\Gamma_{ij}(G, \{G_i, G_j, H\})$ are flag-transitive geometries if and only if the pregeometry $\Gamma''(G, \{G_1, \ldots, G_n, H\})$ is a flag-transitive geometry.

Proof. The implication from right to left is obvious since the truncations of a flag-transitive geometry are flag-transitive. For the converse, we proceed by induction. The result holds for $n = 2$ by lemma 2.9.4. If the result is true for $n - 1$, it remains to show that

$$(A_1 \cap \ldots \cap A_{n-1} \cap H) \cap A_n \supset A_1 A_n \cap \ldots \cap A_{n-1} A_n \cap HA_n \quad (2.9.3)$$

The left part of equation 2.9.3 can be written as

$$(A_1 \cap \ldots \cap A_{n-1} \cap H) \cap A_n = ((A_1 \cap H) \cap \ldots \cap (A_{n-1} \cap H)) (A_n \cap H) A_n \quad (2.9.4)$$

because $(A_n \cap H) A_n = A_n$. Since $\Gamma'$ is flag-transitive by assumption, one obtains that the right member of equation 2.9.4 is equal to

$$((A_1 \cap H) (A_n \cap H) \cap \ldots \cap (A_{n-1} \cap H) (A_n \cap H)) A_n \quad (2.9.5)$$

Now the $\Gamma_{ij}$'s are flag-transitive, so 2.9.5 is equal to

$$(H \cap A_1 A_n \cap \ldots \cap A_{n-1} A_n) A_n \quad (2.9.6)$$

So we want to see that the right member of equation 2.9.3 is included in 2.9.6. Let $t$ be an element of the right member of equation 2.9.3. One has $t = h \alpha$ for some $\alpha \in A_n$, $h \in H$. Moreover, $t \in A_i A_n$ for every $i \in \{1, \ldots, n-1\}$. Therefore $h = t \alpha^{-1}$ where $\alpha^{-1} \in A_n$ and so $h \in A_i A_n$ where $i \in \{1, \ldots, n-1\}$. This implies that $t$ belongs to 2.9.6. \hfill $\square$

2.9.2 Firmness

We introduced firmness in Definition 2.2.2. We now give a equivalent property in coset geometries.

Let $\Gamma(G, \{G_i \mid i \in I\})$ be a coset geometry. For every $j \in I$, the stabilizer in $G$ of a flag $F_j = \{G_i \mid i \in I - \{j\}\}$ is the subgroup $H_j = \cap_{i \in I - \{j\}} G_i$ and the stabilizer of $G_j$ in $H_j$ is the Borel subgroup $B$. So the number of chambers containing $F_j$ is equal to the index of $B$ in $H_j$. Hence $\Gamma$ is firm if and only if $|H_j|/|B| \geq 2$ for every $j \in I$. 


2.9 Properties of a coset geometry

2.9.3 Residual connectedness

We already introduced the concept of residual connectedness in Section 2.6.1. The following result gives a necessary and sufficient condition for a coset geometry $\Gamma$ of rank $n$ to be residually connected. Moreover, remember that strong connectedness and residual connectedness imply each other in finite rank geometries (see proposition 2.6.3).

**Lemma 2.9.6.** (Buekenhout, Hermand [34]) Let $\Gamma(G, \{G_i \mid i \in I\})$ be a flag-transitive geometry of finite rank greater or equal to 2. Then

1. $\Gamma$ is connected if and only if $G$ is generated by the $G_i$’s.
2. $\Gamma$ is residually connected if and only if for every $J \subset I$ such that $|J| \leq n - 2$, the subgroup $\cap_{j \in J} G_j$ is generated by

$$ \bigcup_{k \in I - J} \left( \bigcap_{j \in J \cup \{k\}} G_j \right) $$

**Proof.** Let $\Gamma$ be connected and let $g \in G$. Let us consider the elements $x = G_i$ and $y = G_jg$. There exists a path

$$ x = G_i * G_1x_1 * G_2x_2 * \ldots * G_nx_n * G_jg = y. $$

By definition of adjacency, we can choose $x_1$ in $G_i$, $x_2$ in $G_1G_i$, ..., $x_n$ in $G_{n-1} \ldots G_1G_i$, and $g$ in $G_n \ldots G_1G_i$. Consequently the $G_i$’s generate $G$. For the converse, if $G$ is generated by the $G_i$’s and if $g = g_n \ldots g_1g_i$, $g_k \in G_k \forall k \in \{i, 1, \ldots, n\}$, then $x = G_i$ and $y = G_jg$ are linked by the path

$$ x * G_1g_i * G_2g_1g_i * \ldots * G_ng_{n-1} \ldots g_1g_i * y. $$

Since $\Gamma$ is flag-transitive, any two elements are linked by a path.

For the second point, it suffices to apply the first point to every residue.

2.9.4 Primitivity

We refer the reader to Cameron [20] for the concept of primitivity in permutation groups. Primitivity in the framework of geometries is studied for instance in Buekenhout, et al. [15].
We recall that an action of group $G$ on a set $\Omega$ is *imprimitive* if it preserves a nontrivial partition of $\Omega$ in congruence classes. The action is called *primitive* otherwise.

Let $\Gamma = \Gamma(G, \{G_i \mid i \in I\})$ be a coset geometry of rank $n \geq 2$. We say that $\Gamma$ is *primitive*, and we write (Pri) for this, provided that $G$ acts primitively on the elements of each given type in the geometry. Lemma 2.9.7 gives a necessary and sufficient condition to check whether a geometry is primitive. This result is due to Wielandt [60] while the proof is due to the author and is inspired from Cameron (Theorem 1.7 in [20]).

**Lemma 2.9.7.** The action of the group $G$ on the cosets of a given subgroup $G_i$ is primitive if and only if $G_i$ is maximal in $G$.

**Proof.** Suppose $G_i$ is not maximal in $G$, i.e. there exists a subgroup $H$ of $G$ such that $G_i < H < G$. Now we have $\{G_ih \mid h \in H\} = H$. Hence, given $g \in G$, one has $\{G_ihg \mid h \in H\} = Hg$. This means $G$ preserves a nontrivial partition. Consequently the action of $G$ on the cosets of $G_i$ is imprimitive.

For the converse, suppose that the action of $G$ on the cosets of $G_i$ is imprimitive. Let $\Delta$ be a nontrivial congruence class. Consider the stabilizer $G_\Delta$ of $\Delta$. Clearly it is a group strictly contained between $G_i$ and $G$.

Consequently we obtain Corollary 2.9.8 whose proof is trivial.

**Corollary 2.9.8.** A coset geometry $\Gamma(G, \{G_i \mid i \in I\})$ is primitive if and only if each $G_i$ is maximal in $G$.

### 2.9.5 The property (QPri)

Let $\Gamma(G, \{G_i \mid i \in I\})$ be a coset geometry. We call $\Gamma$ *quasiprimitive*, and we write (QPri) for this, provided that every $G_i$ is a quasimaximal subgroup of $G$; namely there is a unique maximal chain of subgroups from $G_i$ to $G$. The axiom (QPri) is introduced in Buekenhout, et al. [15].

### 2.9.6 The properties (RPri), (WPri) and (RWPri)

We refer the reader to Buekenhout, et al. [15] and Leemans [44].
We call a geometry $\Gamma(G, \{G_i \mid i \in I\})$ residually primitive, and we write (RPri) for this, provided that each residue $\Gamma_F$ of a flag $F$ is primitive for the group induced on $\Gamma_F$ by the stabilizer $G_F$ of $F$ in $G$. We call $\Gamma$ weakly primitive, and we write (WPri) for this, provided that there exists some $i \in I$ such that $G$ acts primitively on the set of $i$-elements of $\Gamma$.

We call a geometry $\Gamma(G, \{G_i \mid i \in I\})$ residually weakly primitive, and we write (RWPri) for this, provided that for any $\emptyset \subseteq J \subseteq I$ there exists at least one element $i \in I - J$ such that $G_{J \cup \{i\}}$ is maximal in $G_J$. If $\Gamma$ is flag-transitive, it is equivalent to require that each residue $\Gamma_F$ of a flag $F$ is weakly primitive for the group induced by the stabilizer $G_F$ of $F$ in $G$.

The (RWPri) condition implies that all subgroups of the subgroup lattice are pairwise distinct and that $\bigcap_{j \in J} G_j$ is a maximal subgroup of $\bigcap_{j \in I - \{i\}} G_j$ for all $i \in I$. Arranging the indices in a suitable manner, we may also assume that $\bigcap_{j \in \{1, \ldots, i\}} G_j$ is a maximal subgroup of $\bigcap_{j \in \{1, \ldots, i-1\}} G_j$ for $i = 2, \ldots, n$.

We can now summarize under Lemma 2.9.9 the different implications between the variations around primitivity defined above.

**Lemma 2.9.9.** The following implications hold.

1. (RPri) \Rightarrow (Pri) \Rightarrow (QPri);

2. (RPri) \Rightarrow (RWPri) \Rightarrow (WPri).

**Proof.** Straightforward. \hfill \Box

### 2.9.7 The intersection property (IP)$_2$

Let $\Gamma$ be a rank two geometry in which the elements are called points and lines. We say that $\Gamma$ is a partial linear space provided that two distinct points are incident to at most one common line.

We say that $\Gamma$ satisfies the intersection property on all its rank two residues, and we write (IP)$_2$ for this, provided that every rank two residue of $\Gamma$ is either a generalized digon or a partial linear space.

In Section 3.2, we will see a way to check this property. It is based on an algorithm we develop to determine the gonality of a geometry.

Intersection properties are studied deeply in Pasini [47]. We refer also the reader to Buekenhout [7, 9].
2.9.8 The property \((2T)_1\)

Our reference for this section is Buekenhout [13].

We call a pregeometry \(\Gamma\) locally 2-transitive, and we write \((2T)_1\) to mean this, provided that the stabilizer \(G_F\) of any flag \(F\) of rank \(|I| - 1\) acts 2-transitively on the residue \(\Gamma_F\). Hence checking whether a geometry is \((2T)_1\) amounts to check whether the action of each of the minimal parabolic subgroups of \(\Gamma\) on the cosets of the Borel subgroup is 2-transitive. The next lemma gives a necessary condition for a group to have a 2-transitive action on the cosets of one of its subgroups.

**Lemma 2.9.10.** Let \(G\) be a group and let \(H\) be a subgroup of \(G\). If \(G\) acts 2-transitively on the cosets of \(H\) in \(G\) then \(|G|\) must be divisible by \([G : H].([G : H] - 1)\).

**Proof.** Denoting with \(G_x\) the stabilizer in \(G\) of \(x\), a coset of \(H\) in \(G\), and with \(G(x)\) the orbit of \(x\), one has:

\[
|G| = |G(x)|.|G_x| \\
= [G : H].|G_x| \\
= [G : H].|G_x(y)|.|G_{xy}| \\
= [G : H].([G : H] - 1).|G_{xy}|
\]

by applying twice Lagrange formula and by 2-transitivity.

2.9.9 The properties (SN) and (BSN)

A coset geometry \(\Gamma(G, \{G_i \mid i \in I\})\) has the self-normalizing property, and we write (SN) for this, provided that all parabolic subgroups \(G_i\), \(i \in I\) are self-normalized in \(G\). The geometry \(\Gamma\) has the Borel-self-normalizing property, and we write (BSN) for this, provided that the Borel subgroup \(B\) of \(\Gamma\) is self-normalized in \(G\). Observe that (SN) implies (BSN). Those properties have been studied in De Saedeleer [26].

2.10 The property (Apt)

In order to introduce at best our concept of apartment in incidence geometries, we have to pay a little visit to Tits’ buildings. It will provide a guideline for the rest of this section.
2.10.1 Buildings and $BN$-pairs

Let us first give the definition of a complex in the sense of Tits [59].

**Definition 2.10.1.** *(Complex)* Let $\Delta$ be a set endowed with an order relation $\subset$ (which means inclusion and has to be read ‘is a face of’ or ‘is contained in’). Then we call $\Delta$ a complex provided that the ordered subset of all faces of any given element is isomorphic with the ordered set of all subsets of a set. Any two elements $A, B$ have a greatest lower bound, denoted by $A \cap B$.

A subcomplex of a complex $\Delta$ is defined in the natural way.

Tits [59] defines a building to be a pair $(\Delta, \mathcal{A})$, where $\Delta$ is a complex and $\mathcal{A}$ a set of subcomplexes of $\Delta$ whose elements are called apartments and where the following four conditions hold.

1. **(B1)** $\Delta$ is thick;
2. **(B2)** The elements of $\mathcal{A}$ are thin chamber complexes;
3. **(B3)** Any two elements of $\Delta$ belong to an apartment;
4. **(B4)** If two apartments $\Sigma$ and $\Sigma'$ contain two elements $A, A' \in \Delta$, there exists an isomorphism of $\Sigma$ onto $\Sigma'$ which leaves invariant $A, A'$ and all their faces.

In this definition, ‘thick’ has to be taken in the same sense as in Definition 2.2.2.

The notion of a building leads to the concept of a $BN$-pair of a group [1, 47, 59]. We first give the definition of a $BN$-pair and then the close interaction that buildings and $BN$-pairs share.

**Definition 2.10.2.** *(BN-pair)* Given a group $G$, a $BN$-pair in $G$ is a system $(B, N)$ consisting of two subgroups such that

1. **(BN0)** $\langle B, N \rangle = G$;
2. **(BN1)** $B \cap N =: H \triangleleft N$;
3. **(BN2)** The group $W := N/H$ has a generating set $R$ such that the following two relations hold for any $r \in R$ and any $w \in W$:
   \[ rBwB \subset BwB \cup BrwB; \]
The group $W$ is called the Weyl group of the $BN$-pair. The quadruple $(G, B, N, R)$ is called a Tits system. One also gets that $G = B N B$. This is called the Bruhat decomposition of $G$.

Theorem 2.10.3 (Theorem 6.56 in [1]) emphasizes the link between buildings, apartments and $BN$-pairs. In this theorem, an action of a group $G$ on a building $(\Delta, A)$ is strongly transitive if $G$ acts transitively on the set of pairs $(\Sigma, C)$ consisting of an apartment $\Sigma \in A$ and a chamber $C \in \Sigma$.

Theorem 2.10.3. Given a $BN$-pair in $G$, the generating set $R$ is uniquely determined, and the elements of $W$ have order 2. There is a thick building $\Delta$ that admits a strongly transitive $G$-action such that $B$ is the stabilizer of a fundamental chamber and $N$ stabilizes a fundamental apartment and is transitive on its chambers.

Conversely, suppose that a group $G$ acts strongly transitively on a thick building $\Delta$ with fundamental apartment $\Sigma$ and fundamental chamber $C$. Let $B$ be the stabilizer of $C$ and let $N$ be a subgroup of $G$ that stabilizes $\Sigma$ and is transitive on the chambers of $\Sigma$. Then $(B, N)$ is a $BN$-pair in $G$ and $\Delta$ is determined up to isomorphism.

With this in mind, we want to find an analogue notion of apartments in coset geometries.

### 2.10.2 Apartments in coset geometries

Many mathematicians have worked on the concept of an apartment in incidence geometries. We refer for instance the reader to Buekenhout [9, 17], Heiss [33], Higman, et al. [35], Ronan, et al. [51], Stroth, et al. [55, 54].

We develop here the concept introduced in Buekenhout and Lee-mans [17].

**Definition 2.10.4.** (*Coxeter diagram*) A geometry $\Gamma$ determines a Coxeter diagram $\text{Cox}(\Gamma) = (I, M_{ij})$ where each ordered pair of elements in $I$ provides an integer $M_{ij}$ with the usual constraints: $M_{ij} = M_{ji}$ for all $i, j \in I$, $M_{ii} = 1$ for any $i \in I$ and $M_{ij} \geq 2$ for all $i, j \in I$. The matrix $M$ is called a Coxeter matrix. For each $i \neq j$, we require that any flag residue of $\Gamma$ of type $\{i, j\}$ has gonality $M_{ij}$. 

2.10 The property (Apt)

The idea here is that, given a geometry \( \Gamma \), an apartment in \( \Gamma \) is a thin subgeometry \( \Sigma \) such that \( \text{Cox}(\Sigma) = \text{Cox}(\Gamma) \).

**Definition 2.10.5. (Apartment in coset geometries)** Let \( I = \{1, \ldots, n\} \), let \( \Gamma(G, \{G_i \mid i \in I\}) \) be a coset geometry of rank \( n \). Let \( B \) be its Borel subgroup, i.e. \( B = \cap_{i \in I} G_i \). We say that \( \Gamma \) satisfies the property (Apt) (or that \( \Gamma \) has apartments) if there exists a subgroup \( N \) of \( G \) such that the following three conditions hold:

1. \( \langle B, N \rangle = G \);
2. \( H := N \cap B \) is a normal subgroup of \( N \);
3. the coset geometry \( \Sigma(N, \{N \cap G_i \mid i \in I\}) \) is a thin, residually connected, flag-transitive geometry such that \( \text{Cox}(\Gamma) = \text{Cox}(\Sigma) \).

An apartment in our sense is not a purely combinatorial object. It comes with a group \( N \) acting flag-transitively on it.

**Definition 2.10.6.** Let \( I = \{1, \ldots, n\} \), let \( \Gamma(G, \{G_i \mid i \in I\}) \) be a rank \( n \) coset geometry. We say that \( \Gamma \) satisfies (Apt) \( r \) where \( 2 \leq r \leq n \) if every rank \( r \) residue of \( \Gamma \) satisfies (Apt).

**Remark 2.10.7.** At this point, it seems appropriate to make an observation and a historical remark.

In Building Theory, an apartment is a combinatorial object. In the context of coset geometries, the definition of an apartment that we give involves groups. Therefore it appears legitimate to call an apartment in our sense an apartment with group, or a transitive apartment in order to underline the difference between those two notions, even though they are similar in many ways.

Already in [9], Buekenhout emphasizes this difference. However our terminology is slightly different from his. He defines an apartment \( \mathcal{A} \) in a geometry \( \Gamma \) as a thin subgeometry of \( \Gamma \) which is such that \( \mathcal{A} \) and \( \Gamma \) have the same Coxeter diagram. If \( \Gamma \) has rank 2, then the apartments are the circuits of minimal length \( 2g \) in the incidence graph of \( \Gamma \), where \( g \) is the gonality of \( \Gamma \). This definition is purely combinatorial.

Now given a coset geometry \( \Gamma(G, \{G_i \mid i \in I\}) \), if \( \mathcal{A} \) is an apartment in the sense of Buekenhout containing the maximal flag \( C \) of \( \Gamma \), then \( N \) denotes the stabilizer of \( \mathcal{A} \) in \( \Gamma \) and \( H \) the subgroup \( B \cap N \) where \( B \) is the Borel subgroup of \( \Gamma \). Then Buekenhout sets that \( \mathcal{A} \) is a regular apartment if \( N \) is transitive on the maximal flags of \( \mathcal{A} \). Finally, given a
coset geometry $\Gamma(G, \{G_i \mid i \in I\})$, he sets that $\Gamma$ satisfies (AP) provided that $\Gamma$ has regular apartments, that $G$ acts transitively on them and that the same holds in every residue of a flag of $\Gamma$, for its $G$-stabilizer.

In the remaining of the dissertation, whenever we talk about an apartment, we always emphasize whether we talk about an apartment in the combinatorial sense or in the sense of Definition 2.10.5.

Theorem 2.10.8 gives a powerful condition to check whether a geometry satisfies (Apt).

**Theorem 2.10.8.** If $\Gamma(G, \{G_i \mid i \in I\})$ is a flag-transitive geometry of rank $n := |I|$ satisfying (Apt), then any residue of rank $r \geq 2$ of $\Gamma$ is a geometry satisfying (Apt).

**Proof.** Remark first that it is enough to prove that if $\Gamma$ satisfies (Apt)$_n$ then it satisfies (Apt)$_{n-1}$.

Let $\Gamma(G, \{G_i \mid i \in I\})$ be a coset geometry satisfying (Apt), let $N$, $B$, $H$, $\Sigma$ be as in Definition 2.10.5, let $G_\alpha \in \{G_i \mid i \in I\}$. The subgroup $G_\alpha$ defines a residue $\Gamma_\alpha(G_\alpha, \{G_\alpha \cap G_i \mid i \in I - \alpha\})$. We want to show that $N \cap G_\alpha$ is a subgroup of $G_\alpha$ which satisfies the three properties (P1), (P2) and (P3).

First, note that $B$ is also the Borel subgroup of $\Gamma_\alpha$, i.e. $B \cap G_\alpha = B$. Hence,

$$\langle B, N \cap G_\alpha \rangle = \langle B \cap G_\alpha, N \cap G_\alpha \rangle = \langle B, N \rangle \cap G_\alpha = G \cap G_\alpha = G_\alpha$$

where the second equality holds by residual connectedness (see Theorem 2.9.6). It follows that (P1) is verified.

In order to prove that (P2) is satisfied, we introduce a straightforward result in group theory. Let $N$ be a group, let $H \triangleleft N$ and let $H \leq K \leq N$. Then we have $H \triangleleft K$.

Therefore in order to prove that $H_\alpha := B \cap (N \cap G_\alpha)$ is a normal subgroup of $N \cap G_\alpha$, we only have to prove that $H_\alpha \triangleleft N$. This is almost
immediate:

\[ H_\alpha = B \cap N \cap G_\alpha \]
\[ = B \cap G_\alpha \cap N \]
\[ = B \cap N \]
\[ = H \]

where the third equality holds because \( B \) is a subgroup of \( G_\alpha \). Now \( H \triangleleft N \).

Now define the coset geometry \( \Sigma_\alpha = \Gamma(N \cap G_\alpha, \{ N \cap G_\alpha \cap G_i \}_{i \in I \setminus \{ \alpha \}}) \).
It is clearly a residue of \( \Sigma \). As \( \Sigma \) is thin, residually connected and flag-transitive, so is \( \Sigma_\alpha \). We want to see that

\[ \text{Cox}(\Gamma_\alpha) = \text{Cox}(\Sigma_\alpha) \]

i.e. \( (I \setminus \{ \alpha \}, M^1_{ij}) = (I \setminus \{ \alpha \}, M^2_{ij}) \) following definition 2.10.4. However \( \Gamma_\alpha \) (resp. \( \Sigma_\alpha \)) is a residue of rank \( n - 1 \) of \( \Gamma \) (resp. \( \Sigma \)). Hence, the Coxeter matrix of \( \Gamma_\alpha \) (resp. \( \Sigma_\alpha \)) is the Coxeter matrix of \( \Gamma \) (resp. \( \Sigma \)) in which the line and the column \( \alpha \) have been deleted. The result is now straightforward.

\[ \square \]

Theorem 2.10.8 is a reformulation of Proposition 2.10.9 that appears in [17].

**Proposition 2.10.9.** (Buekenhout, Leemans [17]) *Let \( \Gamma = (G, (G_i)_{i \in I}) \) be a flag-transitive geometry of rank \( n = |I| \). Then \((\text{APT})_{r+1} \Rightarrow (\text{APT})_r\) for all \( r \geq 2 \).*

**2.10.3 Example of a coset geometry satisfying (Apt)**

In order to illustrate our concept of apartment in a coset geometry, we develop an example in this section. This example was the object of a research the author did under the supervision of Dimitri Leemans in 2010 [23].

Let \( q = p^m \) be a prime power and consider the group \( G := \text{AGL}_2(q) \) acting on the affine plane \( \text{AG}(2, q) \cong A(\mathbb{F}_q^2) \). The group \( G \) acts on the points and the lines of the affine plane. The action is 2-transitive on the set of points and transitive imprimitive on the set of lines.
Let $G_p$ be the stabilizer in $G$ of a given point $p$ and let $G_l$ be the stabilizer in $G$ of a given line $l$ through $p$. It is easily seen that $G_p \cong \text{GL}_2(q)$. Less easy is to see that $G_l \cong \text{AGL}_1(q) : \text{AGL}_1(q)$.

We define a rank 2 geometry $\Gamma$ on the set $I = \{P, L\}$ where $P$ is the set of points of the affine plane and $L$ is the set of lines of the affine plane. Incidence is induced by the symmetrized inclusion. This geometry is firm, residually connected and $\text{AGL}_2(q)$ acts transitively on the flags of $\Gamma$. Thanks to Tits’ algorithm, we can define $\Gamma$ as the coset geometry $\Gamma(G, \{G_p, G_l\})$. The geometry $\Gamma$ belongs to the diagram given in Figure 2.3. The Borel subgroup of $\Gamma$ is $B \cong \text{GL}_1(q) : \text{AGL}_1(q)$.

Since the gonality is 3, the apartments in the combinatorial sense are triangles.

**Theorem 2.10.10.** There are subgroups of $\text{AGL}_2(q)$, isomorphic to $D_6$, that satisfy conditions (P1), (P2) and (P3). All subgroups of $\text{AGL}_2(q)$ that satisfy (P1), (P2) and (P3) are isomorphic to $D_6$.

**Proof.** The Borel subgroup $B$ is the stabilizer of the point $p$ and the line $l$. On $l$, the group $B$ has two orbits. One is $O_1 := \{p\}$, the other is $O_2 := l - p$, i.e. the line $l$ without the point $p$. Now let $p_1 \neq p$ be a point on the line $l$ and let $p_2$ be a point not on $l$. Now we want to determine the stabilizer of the unordered triple $\{p, p_1, p_2\}$ of noncollinear points.

Let us call such a triple a triangle and let $\Omega$ be the set of triangles of the affine plane. We have

$$|\Omega| = \frac{q^2(q^2 - 1)(q^2 - q)}{6} \quad (2.10.1)$$

Indeed, choosing a triangle amounts to choose any two distinct points in a first place and then a third point that is noncollinear with the first two. There are $q^2(q^2 - 1)$ ways to choose the first two points taking the order into consideration and $q^2 - q$ ways to choose the third point. Since
2.10 The property (Apt)

\[ \text{Figure 2.4: The diagram of } \Sigma(N, \{N_1, N_2\}) \]

the order in which we choose the points does not matter, we divide the product \( q^2(q^2 - 1)(q^2 - q) \) by 6 to obtain the size of \( \Omega \). The group \( \text{AGL}_2(q) = G \) acts as a transitive permutation group on \( \Omega \) since \( G \) is transitive on the affine bases of \( \text{AG}(2, q) \). The stabilizer \( G_\alpha \) of a triangle \( \alpha \in \Omega \) satisfies equation 2.10.2 by virtue of Lagrange theorem.

\[ |G_\alpha| = \frac{|G|}{|\Omega|} \tag{2.10.2} \]

We reformulate equation 2.10.2 as in equation 2.10.3.

\[ |G_\alpha| = \frac{q^2(q^2 - 1)(q^2 - q)}{q^2(q^2 - 1)(q^2 - q)} = 6 \tag{2.10.3} \]

Hence the stabilizer of \( \{p, p_1, p_2\} \) is a subgroup of \( \text{AGL}_2(q) \) isomorphic to \( S_3 \cong D_6 \).

Let \( N \) be this stabilizer. In particular, there exists an element of \( N \) that fixes \( p_2 \) and that exchanges \( p \) and \( p_1 \). Hence \( N \) stabilizes \( l \); the point \( p \) is sent in \( O_2 \) and \( p_1 \) in \( O_1 \). It follows that \( G_l \) is a subgroup of \( \langle B, N \rangle \).

Define the geometry \( \Xi(G, \{G_p, G_l, N\}) \). We leave the reader check for himself that \( \Xi \) is firm, residually connected and flag-transitive. By residual connectedness, \( \langle G_l, N \rangle = G \) and thus \( \langle B, N \rangle = G \). Consequently (P1) is satisfied.

One verifies easily that \( B \cap N \), the Borel subgroup of \( \Sigma \), is trivial and thus normal in \( N \). Therefore (P2) is satisfied.

In order to check that (P3) is verified, observe that \( N_1 := G_p \cap N \) and \( N_2 := G_l \cap N \) are both isomorphic to \( C_2 \). Now define \( \Sigma(N, \{N_1, N_2\}) \). We leave to the reader check for himself that \( \Sigma \) belongs to the diagram of Figure 2.4.

By virtue of the developments made to determine the order of \( \Omega \), it follows that the only subgroups of \( G \) that satisfy (P1), (P2) and (P3) are isomorphic to \( D_6 \). 

\[ \square \]
As a direct consequence of Theorem 2.10.10 we have Corollary 2.10.11.

**Corollary 2.10.11.** Using the notation of Theorem 2.10.10, the geometry $\Gamma(G, \{G_p, G_l\})$ satisfies (APT).

*Proof.* The apartments are the subgeometries $\Sigma(D_6, \{C_2, C_2\})$ as they are described in Theorem 2.10.10. \qed
Chapter 3

A new algorithm to find the gonality of a coset geometry

This chapter focuses on the gonality of a geometry. We present an algorithmic approach to gonality in coset geometries that permitted to obtain original results with help of Magma [3]. An implementation in Magma code of the algorithm we develop is given in Chapter 6.

Chapter 3 is part of a preprint from Leemans and the author. It was submitted for publication [24].

3.1 Motivation

Given a rank 2 geometry $\Gamma$ whose gonality is $g$, a combinatorial apartment is a circuit of length $2g$. Taking Remark 2.10.7 into consideration, we see that gonality is thus linked very closely to apartments. In particular, a good question for further study of the concept of an apartment in a coset geometries would be: doesn’t any circuit deserve the name apartment, i.e. is our concept of apartment too strong? It is thus very natural to study gonality while studying apartments.

In Magma [3], a function is implemented that permits to calculate the diagram of a geometry. This function is called Diagram. It is very fast on small geometries, though it requires much memory for coset geometries with a large group and small maximal parabolic subgroups. That is why Leemans [44] could not determine the gonality nor the diameters of two rank 2 geometries for $M_{24}$. Those are the rank 2 geometries for $M_{24}$ labelled 2.8 and 2.11 in [44]. He was therefore unable
to look for apartments in those geometries since he needed the Coxeter diagrams.

We take an approach which is different from Diagram. This approach produces an algorithm that calculates the gonality of a geometry as well as the diameters. It is slower than Diagram, but the memory used is very reasonable in comparison. Thanks to it we managed to calculate the two unknown diagrams of Leemans [44]. They are given in Figure 3.1.

### 3.2 An algorithm to compute gonality

Let $\Gamma(G, \{G_0, G_1\})$ be a rank two coset geometry. Define the set

\[ s_1 := \{xy \mid x \in G_0, y \in G_1\}. \]

It covers exactly the cosets of $G_1$ that are incident to $G_0$. At each step $k$, define $s_k$ recursively as

\[ s_k := \{xy \mid x \in s_{k-1}, y \in G_i, i = k \pmod{2}\}. \]

If we assume that each element at step $k$ is connected in a unique way to the starting element $G_0$, then, the size of the set $s_k$ must be

\[
\begin{align*}
\left(1 + \sum_{i=1}^{k/2} \alpha(\beta - 1)^i(\alpha - 1)^{i-1}\right) \cdot |G_0| \\
\left(\sum_{i=0}^{(k-1)/2} \alpha(\beta - 1)^i(\alpha - 1)^i\right) \cdot |G_1|
\end{align*}
\]

for $k$ even and

for $k$ odd, where $\alpha = [G_0 : G_0 \cap G_1]$ and $\beta = [G_1 : G_0 \cap G_1]$. Let us denote this number by $f(k)$.

**Lemma 3.2.1.** The gonality of $\Gamma$ is the smallest integer $k$ such that $|s_k| \neq f(k)$.

**Proof.** It suffices to observe that $f(k)$, by definition, gives the number of elements of type $k \pmod{2}$ that are at distance $k$ of $G_0$ provided there is a unique shortest path of length $k$ from $G_0$ to each of them. Once $f(k) \neq |s_k|$, there must be an element $X$ at distance $k$ of $G_0$ and two distinct paths of length $k$ from $G_0$ to $X$. \qed
2.8
\[
\begin{array}{cccc}
0 & 23 & 6 & 23 \\
\circ & \cdots & \circ & 1 \\
1 & 6 \\
1,457,280 & 5,100,480 \\
L_3(2) & [S_4] \times 2
\end{array}
\]
\[
\text{Aut}_I(\Gamma) = M_{24} \\
B = S_4
\]

2.11
\[
\begin{array}{cccc}
0 & 16 & 7 & 16 \\
\circ & \cdots & \circ & 1 \\
2 & 6 \\
1,457,280 & 3,400,320 \\
L_3(2) & [A_4] : S_3
\end{array}
\]
\[
\text{Aut}_I(\Gamma) = M_{24} \\
B = S_4
\]

Figure 3.1: Diagrams, unknown so far

As an immediate corollary, we have the following lemma.

**Lemma 3.2.2.** The following statements are equivalent.

1. \(|s_2| = (\alpha(\beta - 1) + 1)|G_0|;\)

2. \(g_{01} > 2.\)

Observe now that a rank 2 geometry satisfies (IP)\(_2\) if and only if it satisfies the Lemma 3.2.2. In particular, the second point is exactly the intersection property (IP)\(_2\).

Writing a program that finds the smallest \(k\) as in Lemma 3.2 is very easy though this algorithm takes more time than diagram. The advantage is that it does not use much memory. We give an implementation of such an algorithm in Chapter 6. Using the same ideas, we can determine the diameters \(d_{01}\) and \(d_{10}\). Indeed, \(d_{ij}\) is the smallest integer \(k\) such that \(|s_k| = |G|\). We may obtain \(d_{ji}\) by inverting \(G_0\) and \(G_1\).
A new algorithm to find the gonality of a coset geometry
Chapter 4

New algorithms to find apartments in coset geometries

In this chapter, we present new algorithms to find apartments in coset geometries. They permit to answer to some questions left open in a paper by Buekenhout and Leemans [17]. In particular, one of our algorithms is very efficient. We compare the performance of this algorithm to the one given by Buekenhout and Leemans. This chapter is part of a preprint by Leemans and the author submitted for publication [24].

4.1 The Buekenhout-Leemans algorithm

So far, the algorithm we have used to check whether a geometry satisfies (Apt) has been the algorithm described in Buekenhout and Leemans [17]. Let us call this algorithm Apartment. We recall it in Figure 4.1. With Apartment, many results were produced. However, it is a kind of brute force algorithm and, as it is shown in Section 4.5, Apartment is very greedy in time.

4.2 Open questions

The algorithm Apartment was implemented in MAGMA [3] and tested on the 420 geometries available in Leemans [44]. Already it appeared that some geometries do not satisfy (Apt). However, for some geometries,
Compute the Coxeter diagram of the geometry \( \Gamma_1 \)
for each subgroup \( H \) of \( B \)
let \( N_H \) be the normalizer in \( G \) of \( H \)
for each subgroup \( N \) of \( N_H \)
compute the coset geometry \( \Gamma_2 \)
if this geometry is thin, RC, FT and has same Coxeter
diagram as \( \Gamma_1 \)
return true
return false

Figure 4.1: The Buekenhout-Leemans algorithm

the calculations could not terminate. This happened for two reasons. First \textit{Apartment} is a greedy algorithm; therefore memory was sometimes insufficient and sometimes calculations were taking too long. The other reason is that two geometries in Leemans [44] had an unknown diagram. The function implemented in \textsc{Magma} [3] to compute the diagram of a geometry sometimes takes too much memory when the geometry is very large. However, \textit{Apartment} only requires the gonality of a geometry. That is why we worked out an algorithm which calculates only the gonality. It is the algorithm described in Chapter 3. Since this algorithm is able to calculate the two unknown diagrams we mentioned, we could try the algorithm \textit{HasApartment} on them and we obtained an answer for each of them (see Table 4.2).

It appears that 157 geometries out of the set of 420 geometries from [44] satisfy \((\text{Apt})\). Also, many geometries have at least two different kinds of apartments in our sense.

### 4.3 Two new algorithms

In Section 4.3.1, we present a new algorithm to check whether a geometry satisfies \((\text{Apt})\). This algorithm is based on a new approach to the problem. In section 4.3.2, we present an improvement of this same algorithm, based on Proposition 2.10.9 (or equivalently Theorem 2.10.8), that sometimes gives a faster answer on high rank geometries.
4.3 Two new algorithms

4.3.1 A new approach

In this section, we present a new approach to find apartments in a coset geometry. Let us begin with a coset geometry $\Gamma(G, \{G_i \mid i \in I\})$ of rank $n := |I|$. As usual, we denote by $B$ the Borel subgroup of $\Gamma$.

In order to find apartments, one has to find subgroups $N < G$ such that the following three conditions of Definition 2.10.5 hold:

1. $\langle B, N \rangle = G$;
2. $B \cap N =: H \trianglelefteq N$; and
3. $\Sigma(G \cap N, (G_i \cap N)_{i \in I})$ is a geometry which is thin, residually connected and flag-transitive.

Our strategy is to build apartments from the bottom rather than from the top. We consider any subgroup $H$ of the Borel subgroup $B$ of $\Gamma$ as a potential Borel subgroup of $\Sigma$. Given such a subgroup $H \leq B$, we build the minimal parabolic subgroups of $\Sigma$ in the following way.

We first compute the normalizer $N_{P_i}(H)$ of $H$ in each minimal parabolic subgroup $P_i$ of $\Gamma$. Then we consider the quotient $N_{P_i}(H)/H$ and the natural homomorphism $\varphi_i : N_{P_i}(H) \to N_{P_i}(H)/H$. Now define the set

$$S_i := \{\rho \in N_{P_i}(H) \mid (\varphi_i(\rho))^2 = 1\}$$

of elements of $N_{P_i}(H)$ each of which has an image by $\varphi_i$ in $N_{P_i}(H)/H$ that is an involution. We then have Lemma 4.3.1.

**Lemma 4.3.1.** Under the previous hypotheses, $[\langle H, \rho \rangle : H] = 2$ for any $\rho \in S_i$.

**Proof.** Since $(\varphi_i(\rho))^2 = 1$, one has $H\rho^2 = H$ and since $\rho \in N_{P_i}(H)$, one has $\rho H = H\rho$. Now take $x \in \langle H, \rho \rangle$. One can write $x = h_1\rho^{p_1} \ldots h_k\rho^{p_k}$ for some $h_i \in H$ and some powers $p_i \in \mathbb{Z}$, where $i \in \{1, \ldots, k\}$. Let us consider $Hx = Hh_1\rho^{p_1} \ldots h_k\rho^{p_k}$. Using the fact that $\rho$ normalizes $H$, one gets $Hh_1\rho^{p_1} \ldots h_k\rho^{p_k} = H\rho^{p_1} \ldots \rho^{p_k}$. Using the remarks made at the beginning of the proof, one concludes easily that $Hx = H$ or $Hx = H\rho$, i.e. there are only two cosets of $H$ in $\langle H, \rho \rangle$. \qed

We now require the minimal parabolic subgroups of $\Sigma$ to be the subgroups $\langle H, \rho_i \rangle$ where the elements $\rho_i \in S_i$ (with $i \in \{1, \ldots, n\}$) are chosen according to some conditions we develop below. This ensures
that $\Sigma$ is thin and residually connected. Clearly, we also have $H < \langle H, \rho_1, \ldots, \rho_n \rangle$ where $\rho_i \in S_i$.

Since we want $\Sigma$ to belong to the same Coxeter diagram as $\Gamma$, we need to choose $\rho_i \in S_i$ such that for any $i, j \in \{1, \ldots, n\}$, $i \neq j$, one has $\langle H, \rho_i, \rho_j \rangle / H \cong D_{2g_{ij}}$ where $D_{2m}$ is the dihedral group of order $2m$. Indeed, $\langle H, \rho_i, \rho_j \rangle / H$ is a group generated by the two involutions $x := \varphi_i(\rho_i)$ and $y := \varphi_j(\rho_j)$. Hence,

$$\langle H, \rho_i, \rho_j \rangle / H = \langle x, y \mid x^2 = y^2 = (xy)^g = 1 \rangle$$

for some $g \in \mathbb{N}$. Requiring $\langle H, \rho_i, \rho_j \rangle / H \cong D_{2g_{ij}}$ is thus the same as requiring $g$ to be equal to the Schläfli symbol $g_{ij}$ of the types $i$ and $j$ in $\Gamma$. Moreover, in order to verify this, it is sufficient to test if $|\langle H, \rho_i, \rho_j \rangle| = 2g_{ij}|H|$ because this implies that $\langle H, \rho_i, \rho_j \rangle / H$ is of order $2g_{ij}$. Using the same argument as before, this group could then only be the dihedral group of order $2g_{ij}$.

Let us write $N = \langle H, \rho_1, \ldots, \rho_n \rangle$ and $N_i = \langle H, \rho_1, \ldots, \hat{\rho}_i, \ldots, \rho_n \rangle$. If $\langle B, N \rangle = G$, then for each $n$-uple of elements $\rho_i \in S_i$ satisfying the conditions above, we build the coset geometry $\Sigma(N, \{N_i \mid i \in I\})$. By construction, $\Sigma$ is thin. It remains to check residual connectedness and flag-transitivity to ensure that $\Sigma$ gives us an apartment.

Lemma 4.3.2 gives a sufficient condition to discard subgroups of the Borel subgroup for which there is no chance to find an apartment.

**Lemma 4.3.2.** Let $\Gamma$ be a coset geometry for the group $G$, and let $\Sigma$ be an apartment of $\Gamma$ with Borel subgroup $H$. Let $S$ be the set of the Schläfli symbols (including the number 2) of the diagram of $\Gamma$. Then we have

$$|N_G(H)| \equiv 0 \mod(2 \text{LCM}(S), |H|) \quad (4.3.1)$$

**Proof.** Let $M$ be the Coxeter matrix associated to $Cox(\Gamma)$. It suffices to observe that if $\Sigma(N, \{N_i \mid i \in I\})$ is an apartment with Borel subgroup $H$, the quotient $N / H$ must have order divisible by each of the entries of $M$ multiplied by 2. \qed

This lemma gives a condition that the Buekenhout-Leemans algorithm does not include. We implemented a version of this algorithm in which we took it in consideration and we obtained a faster algorithm, although it is still slower than the algorithm *HasApartment* described in Figure 4.2. We give in Table 4.4 the time comparison between the three algorithms.
for each class of subgroups of $B$ take a representative $K$
if condition of lemma 4.2 is satisfied then
for each subgroup $H$ in the class of $K$ do
for each minimal parabolic $P$ of $\Gamma$
compute the normalizer of $H$ in $P$
find involutions in it and take their preimages
for each good nuple of such elements
compute $\Sigma$ as above
if it is T, RC, FT return true
return false

Figure 4.2: The algorithm $\text{HasApartment}$

In view of the previous discussion, we give in Figure 4.2 our new algorithm. An implementation in Magma code is given in Chapter 6.

**Remark 4.3.3.** It is natural to wonder whether it is necessary to consider elements of the minimal parabolic subgroups as in Lemma 4.3.1. Isn’t it enough to consider involutions in the minimal parabolic subgroups that are not in the Borel subgroup? In view of Lemma 4.3.1, the answer is no. However, taking this approach into consideration is very interesting. Consider the algorithm described in Figure 4.2 that we modify as follows: for each minimal parabolic subgroup, take the involutions that are not in the Borel subgroup. If such involutions do not exist then proceed as is described in Figure 4.2. An implementation of this algorithm is given in Chapter 6. The resulting algorithm - which we call $\text{HasApInv}$ for the rest of this chapter - is much faster than $\text{HasApartment}$ as is shown in Figure 4.5. This follows from the fact that there are generally far less elements $\rho$ to consider. However $\text{HasApInv}$ is not deterministic in the following sense: if it returns ‘true’ then it answers correctly; but if it answers ‘false’, the correct answer might still be ‘true’. This happens for some rank 2 geometries for $M_{24}$. Fortunately, it appears that in more than 99% of the geometries considered in this chapter, the correct answer is always given by $\text{HasApInv}$.

Considering that some groups may have a very large order, we prove in Lemma 4.3.4 a fast way to find involutions. We did not include it in the implementations of our algorithms given in Chapter 6; it can be easily included when working with a group containing a very large number of involutions.
Suppose $G$ has only one conjugacy class of subgroups of order 2. Let $L$ be a subgroup of this class. Consider $N = N_G(L)$, i.e.

$$N = \{g \in G \mid gL = Lg\}$$

Now define $C_N = \{Ng \mid g \in G\}$ the set of right cosets of $N$ in $G$. Pick one $g$ in $G$ for each coset of $N$, i.e. a minimal subset $T$ of $G$ such that $C_N = \{Nt \mid t \in T\}$.

**Lemma 4.3.4.** With our previous notations and hypotheses, the set

$$I = \{i^t \mid i \in L, i \neq id_G, t \in T\}$$

is the set of involutions of $G$, where $i^t = tit^{-1}$.

**Proof.** First we prove $I$ is a set of involutions. Let $y \in I$. There exists $t \in T$ such that $y = i^t$, where $i \in L$, $i \neq id_G$. So we have

$$y^2 = (i^t)^2$$

$$= (tit^{-1})(tit^{-1})$$

$$= tit^{-1}$$

$$= tt^{-1}$$

$$= id_G$$

Now we want to see that every involution of $G$ is contained in $I$.

If $g$ and $h$ are conjugate in $G$, then there exists $x$ in $G$ such that $g^x = h$. We have for every $y \in C_G(g)$ that $g^y = g$ implies that $g^{yx} = g^x$. Hence $yx$ sends $g$ onto $h$.

Now set $H := C_G(g)$. We have that $Hx$ is the set of elements which send $g$ onto $h$. Moreover there exists $t \in T$ such that $t \in Hx$ because $T$ contains one representant of each right coset of $H$. Therefore $g^t = h$ and so $h \in I$.

The above proof is due to the author.

### 4.3.2 An application of Theorem 2.10.8

By Theorem 2.10.8, if one suspects a high rank geometry not to have apartments, it is worth one’s while to check whether its low rank residues have apartments. Indeed, it is intuitive that the higher the rank of a
4.4 Results obtained thanks to the algorithms

for residues R of cg growing in size
    if R does not satisfy Apt return false
return true

Figure 4.3: The algorithm HasApartmentRes

geometry is, the longest a calculation is expected to run. If any residue

 does not satisfy (Apt), then the geometry itself has no apartment.

This allows us to write the algorithm HasApartmentRes (cg is a coset

geometry) given in Figure 4.3.

This approach is generally much faster if the geometry has no apart-

ment. However if the geometry has apartments, checking all of its

residues will be a loss of time. An advantageous strategy is therefore to

first run the algorithm HasApartment and if after some time it did not

give any answer, to run HasApartmentRes.

4.4 Results obtained thanks to the alg-

orithms

Tables 4.1 and 4.2 are taken from Buekenhout and Leemans [17] and are

completed here. Table 4.1 gives, for the nine smallest sporadic groups,

the number of BCDL-geometries of each rank and how much of them

satisfy (Apt), for r = 1, ..., 5. Note that none of the geometries ap-

pearing in Leemans [44] is of rank higher than six and no geometry of

rank six satisfies (Apt). Table 2 gives the references of the geome-

tries, according to [44], that satisfy (Apt). By convention, a geometry

satisfying (Apt)₃ for instance is not mentioned in the (Apt)₂ column

(see Theorem 2.10.8 for the reason why it makes sense). The bold face

numbers are the new results (except for M₁₂ for which there was a typo

in [17] that is now corrected).
<table>
<thead>
<tr>
<th>Group</th>
<th>Rk</th>
<th>#geo.</th>
<th># (Apt)$_2$</th>
<th># (Apt)$_3$</th>
<th># (Apt)$_4$</th>
<th># (Apt)$_5$</th>
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<td></td>
<td>4</td>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>9</td>
<td>8</td>
<td>7</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
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<td>7</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>5</td>
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<td>6</td>
<td>4</td>
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</tr>
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<td>4</td>
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<td>0</td>
</tr>
<tr>
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<td>11</td>
<td></td>
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<td></td>
</tr>
<tr>
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<td>$J_1$</td>
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<td>8</td>
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<td></td>
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<tr>
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<td></td>
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</tr>
<tr>
<td>$HS$</td>
<td>2</td>
<td>6</td>
<td>5</td>
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<td></td>
<td></td>
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<td>6</td>
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</tr>
</tbody>
</table>

Table 4.1: Property $(Apt)_r$ tested on sporadic geometries
### Table 4.2: Census of sporadic geometries satisfying \((\mathsf{Apt}),_r\) (1)

<table>
<thead>
<tr>
<th>(G)</th>
<th>(\text{Rk})</th>
<th>((\mathsf{Apt}),_2)</th>
<th>((\mathsf{Apt}),_3)</th>
<th>((\mathsf{Apt}),_4)</th>
<th>((\mathsf{Apt}),_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_{11})</td>
<td>2</td>
<td>all</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1, 2, 4, 5, 7</td>
<td>3, 6, 9, 10, 11, 12, 13</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
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<td>5, 11</td>
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<td></td>
</tr>
<tr>
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<td>all but 9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>3</td>
<td>6, 9, 10, 14, 16</td>
<td>1 to 5, 7, 8, 11, 12, 13, 15, 17</td>
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<td></td>
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<tr>
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<td>13, 16, 18</td>
<td>1, 2, 5, 7</td>
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</tr>
<tr>
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<td>1</td>
<td>none</td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
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<td></td>
<td>4</td>
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<td>2, 10, 11, 1 to 4</td>
<td>3, 4, 15, 16</td>
<td>none</td>
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<td>5</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(M_{23})</td>
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<td>all</td>
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</tr>
<tr>
<td></td>
<td>3</td>
<td>all</td>
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<td>5</td>
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<td></td>
<td>4</td>
<td>none</td>
</tr>
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<td>(M_{24})</td>
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<td>1 to 6, 8, 9, 10, 12, 14</td>
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<td>1, 2, 4, 5, 6, 11, 20, 21, 22</td>
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<td>1 to 4</td>
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<td>4</td>
<td>all</td>
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<td>8</td>
<td>1 to 4</td>
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<td></td>
<td>6</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>(J_1)</td>
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<td>3</td>
<td></td>
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<td></td>
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<tr>
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<td></td>
<td>1, 2</td>
<td>none</td>
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</table>

4.4 Results obtained thanks to the algorithms
54 New algorithms to find apartments in coset geometries

Table 4.3: Census of sporadic geometries satisfying \((Apt)_r\) (2)

<table>
<thead>
<tr>
<th>G</th>
<th>Rk</th>
<th>((Apt)_2)</th>
<th>((Apt)_3)</th>
<th>((Apt)_4)</th>
<th>((Apt)_5)</th>
</tr>
</thead>
<tbody>
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<td>all but 3</td>
<td>3, 5, 9, 10,</td>
<td>1, 2</td>
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<td></td>
<td>1, 4, 8, 12, 13, 15, 17</td>
<td>11, 16, 18, 19</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(J_3)</td>
<td>2</td>
<td>1, 3, 5, 6, 7</td>
<td>2, 6, 16</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1, 2, 6, 7, 14, 16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(HS)</td>
<td>2</td>
<td>1 to 5</td>
<td>1, 2, 3, 7, 9 to 19</td>
<td>1, 8, 9, 10</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 to 21, 24 to 30</td>
<td>25, 26, 27, 29, 30</td>
<td>39, 40</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td>1, 3, 5, 8 to</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>11, 39, 40</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>4</td>
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<td>5</td>
<td>1, 4, 6, 7, 8</td>
<td>none</td>
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</tr>
</tbody>
</table>

4.5 Time comparison between the algorithms

Table 4.4 gives the time taken by the following four algorithms to test \((Apt)\) on the list of geometries of a given rank for a given group:

- the Bukenhout-Leemans algorithm \textit{Apartment};
- the algorithm \textit{Apartment2}, namely the Bukenhout-Leemans algorithm in which we took into account the condition stated in Lemma 4.3.2;
- our algorithm \textit{HasApartment};
- the variation \textit{HasApInv} that we mentioned in Remark 4.3.3.

The programs stop running on a geometry as soon as an apartment is found (that is, the variable \textit{StopAfterFound} in the implementation of the algorithms is set to be \texttt{true}; see Chapter 6). The symbol \(\infty\) means that we let the program run for more than a week (we sometimes let it run for more than a month) and that it couldn’t terminate in that time. Remember that a week is equal to 604,800 seconds. The computations
were made with a computer having 96 Gb of memory and two 6-core processors running at 3.4 Ghz.

Let us first remark that \texttt{HasApInv} is astonishingly faster than any other algorithm. Considering that it answered wrongly only on 3 geometries out of the 420 in Leemans [44], we realize that it could be advantageous to first try it before any other algorithm. If it answers ‘true’, it would have done it much faster than the others, and if it answers ‘false’, then it is very tempting to believe the answer is truly false. Anyway the time spent would be a fraction of the time spent by any other algorithm.

We observe also that in most cases, \texttt{HasApartment} is much faster than the algorithms \texttt{Apartment} and \texttt{ApartmentBis}. The only exception is for the rank 2 geometries of the group $M_{23}$. However, one comment should be done about it. In fact, the algorithm ran for about 28,500 seconds on the first geometry and then finished running on the remaining 3 geometries very faster. On the same geometries, \texttt{Apartment} took about the same amount of time for each of them.

Another observation is that for each list of geometries for a group, the algorithm \texttt{HasApartment} tends to be faster on high rank geometries than on low rank geometries. This is not the case for \texttt{Apartment} and \texttt{ApartmentBis}. In those cases, our algorithm can sometimes be 10,000 times faster than \texttt{Apartment}.

Finally, we can conclude that \texttt{HasApartment} improves the Buekenhout-Leemans algorithm, even if we compute it taking Lemma 4.3.2 into account.
Table 4.4: Time comparisons of algorithms: execution times in seconds
Chapter 5

Geometries for the sporadic simple group of Suzuki

In this chapter we present some research we made about coset geometries for the sporadic simple group $Suz$. Most of those geometries have been known for more than two decades. The most recent one presented here was discovered some years ago by Leemans and is detailed in [43, 44].

5.1 About the sporadic simple group $Suz$

The existence of the group $Suz$ was first discovered and proved by Michio Suzuki [56] in a paper from 1969. It has order $448,345,497,600$ which makes it the thirteenth biggest sporadic group in terms of the number of elements, just after the group $O’N$ which is larger by only a few billions elements. The group $Suz$ was rediscovered when mathematicians studied the Leech lattice which was found by Witt in 1940 and rediscovered in 1965 by Leech. We refer the reader to Griess [31].

Making calculations with $Suz$ is not easy for several quite obvious reasons. First its size is impressive and its smallest representation as the automorphism group of a graph is on 1,782 points. On the other hand, its subgroup lattice is very big. Indeed, $Suz$ has 6,381 conjugacy classes of subgroups which implies that brute force algorithms are generally not useful, even with the increasing performance of computers.
5.2 Some geometries of various ranks

In 1979, Buekenhout [7] defined the concept of a diagram over a geometry. Some years later, Buekenhout [9] gave a collection of more than one hundred geometries for some simple groups, including all the sporadic groups. In the same time, Ronan and Stroth [51] published a list of minimal parabolic systems for sporadic groups which provides many geometries. The publication of such lists proved somehow that the concept of diagram as defined in [7] is an interesting way to study the sporadic simple groups.

We now present some geometries of ranks from 2 to 7 for the group Suz. So far there is no classification of the BCDL-geometries for this group. Moreover, the list of geometries we present here is certainly not an exhaustive list of the known geometries for Suz. Not every geometry presented in this section has been fully studied. This would have overtaken the framework of this master’s thesis. We concentrated our research on five geometries; the results of this research are presented in Section 5.3. We built each of those geometries in MAGMA [3] in order to guide our research.

5.2.1 Buekenhout: rank 2, 3 and 4 geometries

In Buekenhout [9], a rank 2 geometry is given over the following diagram.

```
0  8  3  8  1
  1 --------- 279
```

We refer to it as the *Buekenhout rank 2 geometry for Suz*. This geometry arises as a truncation of the Buekenhout-Ronan rank 3 geometry that we introduce in Section 5.2.2.

The following one is a rank 3 geometry on which *Suz* acts flag-transitively. It first appeared in Buekenhout [7] and is constructed thanks to works on polar spaces [6, 16]. This geometry belongs to the diagram
A geometry belonging to such a diagram is called an \textit{extended generalized hexagon}. We refer to this geometry as the \textit{rank 3 Buekenhout geometry for Suz}. We treat it in section 5.3.3.

Also, Buekenhout [9] constructed a rank 4 geometry for \textit{Suz} belonging to the diagram

\begin{center}
\begin{tikzpicture}
\node at (0,0) (c) {c};
\node at (1,0) (1) {1};
\node at (2,0) (4) {4};
\node at (3,0) (4) {4};
\draw (c) -- (1);
\draw (1) -- (4);
\draw (4) -- (c);
\end{tikzpicture}
\end{center}

However, the informations available in [9] contain a mistake so we could not build it in Magma and we did not study it any further during the research time of this master’s thesis.

\subsection*{5.2.2 Buekenhout and Ronan: rank 3 geometry}

Ronan [49] gives rank 3 geometries for \textit{Suz}. He acknowledges Buekenhout for observing the existence of the following one in private communications. It is also mentioned in Buekenhout [9] and in Pasini [47]. It belongs to the following diagram. We give details in Section 5.3.2.

\begin{center}
\begin{tikzpicture}
\node at (0,0) (c) {c};
\node at (1,0) (1) {1};
\node at (2,0) (3) {3};
\node at (3,0) (3) {3};
\draw (c) -- (1);
\draw (1) -- (3);
\draw (3) -- (c);
\end{tikzpicture}
\end{center}

A geometry belonging to such a diagram is called an \textit{extended generalized quadrangle}. We refer to it as the \textit{rank 3 Buekenhout-Ronan geometry for Suz}. This geometry gave birth to a rank 4 geometry for the Conway group \textit{Co$_1$} (see for instance Cohen [22]).
5.2.3 Ronan and Smith: a rank 3 geometry

In [50], Ronan and Smith build some geometries for sporadic groups. Among them is the following for $Suz$.

Such a geometry is called a GAB, i.e. a geometry that is almost a building. A GAB is a geometry in which all rank two residues are generalized polygons. Such geometries have been much studied, for instance in Kantor [37, 38].

A geometric construction of this geometry is provided in Yoshiara [63]. This geometry is referred to as the rank 3 Ronan-Smith geometry for $Suz$. It will deserve special attention in section 5.3.4.

5.2.4 Neumaier and the rectagraphs

In 1982, Neumaier [46] constructed a sequence of five geometries $\Gamma_N(i)$ of increasing rank $i+3$ in bijection with the Suzuki tower ($i = 1, \ldots, 4$). These geometries have diagram

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 2 \\
\end{array}
\]

with automorphism groups $\text{PGL}_2(7)$, $G_2(2)$, $J_2.2$, $G_2(4)$ and $Suz.2$ respectively. However, those geometries are not residually weakly primitive (see Leemans [43]) and we will not treat them any further for the time being.
5.3 Testing the axioms

Figure 5.1: The Leemans rank 6 geometry for $Suz$.

\[ B = 3^{2+4}.2.2.S_4 \]

Figure 5.2: The diagram of the Buekenhout rank 2 geometry

5.2.5 Leemans and the Suzuki tower

In 2005, Leemans [43] used the Suzuki tower to build a rank 6 geometry for $Suz$. It is proved in [43] that it is a $BCDL$-geometry. It belongs to the diagram given in Figure 5.1.

We treat it in section 5.3.5, and we call it the rank 6 Leemans geometry for $Suz$ from now on.

5.3 Testing the axioms

In this section we study the properties of five of the geometries presented in Section 5.2. We built each geometry in MAGMA [3] and we obtained many original results. Moreover we sometimes succeeded in proving the results we got with computer free arguments.
5.3.1 The Buekenhout rank 2 geometry

This geometry first appears in Buekenhout [9]. It is a truncation of the Buekenhout-Ronan rank 3 geometry. Buekenhout gives the characterization of Figure 5.2.

We can draw the boolean lattice for this geometry as in Figure 5.3, with the convention (that we will follow in this thesis) that double lines mean maximal inclusion, while simple lines do not state anything more than inclusion. Those maximal inclusions are proved using the ATLAS [25] easily. Thanks to this, we deduce that the geometry is (RWPri). We can even say more since all the inclusions are maximal: this geometry is (Pri). It is flag-transitive since it is the truncation of a flag-transitive geometry. It is also firm, residually connected and satisfies (IP)\textsuperscript{2} since the gonality is 3.

However, it is not (2T)\textsubscript{1}. Indeed, the order of the Borel subgroup is 69,984. According to Lemma 2.9.10, a necessary condition for this geometry to be (2T)\textsubscript{1} is that 279 divides 69,984 which is not the case.

Moreover, Buekenhout states in [9] that this geometry has no apart- ment and so says MAGMA.

As a last comment, we see that a geometry satisfying (Apt) may possess a truncation that does not satisfy (Apt) since it is the case here: the Buekenhout-Ronan rank 3 geometry satisfies (Apt) – as we demonstrate in the next section – while its \{0,1\}-truncation (which is the Buekenhout rank 2 geometry) does not.

5.3.2 The Buekenhout-Ronan rank 3 geometry

We built the coset geometry \(\Gamma\) using MAGMA [3] and the informations available in Buekenhout [9] that we give in Figure 5.4. They are also detailed in Pasini [47].

The residues \(\Gamma_0\) and \(\Gamma_1\) are geometries over the diagrams given in Figure 5.5 and Figure 5.6. Note also that the truncation of the types 0 and 1 is the geometry presented in Section 5.3.1.

First of all, it has to be mentioned that Buekenhout [9] knew that the geometry \(\Gamma\) has all the properties that we now sum up under the name BCDL and he knew the existence of apartments in this geometry, i.e. he knew that \(\Gamma\) satisfies (AP) in the sense of Remark 2.10.7. We will prove both of those statements. We used MAGMA in order to guide our research. However, we managed to get a computer free proof of
5.3 Testing the axioms

\[ G = \text{Suz} \]
448, 345, 497, 600

22, 880

19, 595, 520

[3^2+4.2.2.S_4].2
139, 968

280

2

\[ B = 3^{2+4}.2.2.S_4 \]
69, 984

Figure 5.3: The boolean lattice related to the Buekenhout rank 2 geometry

(1) \text{Suz}

\[ \Gamma \]
0 \hspace{1cm} c \hspace{1cm} 1 \hspace{1cm} 2

1 \hspace{1cm} 9 \hspace{1cm} 3

\[ 2^5.5.11.13 \hspace{1cm} 2^7.5^2.7.11.13 \hspace{1cm} 2^9.5.7.13 \]

[3]U_4(3).2 \hspace{1cm} [3^{3+4}.2.2].S_4.2 \hspace{1cm} [3^5]M_{11}

\[ B = 3^{5+2}.2^{1+2} \]
\[ \Gamma_2 = (2), \Gamma_0 = (3) \]

Figure 5.4: The Buekenhout rank 3 geometry
those facts.

The first step in building the geometry $\Gamma$ was to draw the boolean lattice in order to construct the coset geometry. This lattice is given in Figure 5.7. For the sake of readability, we only give orders of the subgroups and not the indices (which can anyway be computed easily thanks to the orders).

The maximality in $\text{Suz}$ of the parabolic subgroups $G_0 := [3]U_4(3).2$, $G_1 := [3^{1+4}.2.2]S_4.2$ and $G_2 := [3^5]M_{11}$ is proved using the list of maximal subgroups of the Suzuki group available in Wilson [61]. The other maximal inclusions are proved combining MAGMA [3], the ATLAS [25] and simple algebraic arguments. The number below each group is the order of the corresponding group.

Thanks to this lattice, we can deduce several properties of $\Gamma$. Because $G_0, G_1$ and $G_2$ are maximal subgroups of $\text{Suz}$, $\Gamma$ is (Pri). Moreover, because of the maximality of all the inclusions, we deduce that $\Gamma$ is (RWPri). We also check easily that $\Gamma$ is firm, residually connected and satisfies the intersection property in rank 2. In order to see that $\Gamma$ is (2T)$_1$, let us have a look at the action of the minimal parabolic subgroups on the cosets of the Borel subgroup. First, we have

$$
\begin{array}{ccc}
(2) & M_{11} & \\
0 & c & 1 & B = 3^2.2^{1+2} \\
\circ & \circ & \\
1 & 9 & \\
11 & 55 & M_{10} = [3^2.2^{1+2}].2 \\
\end{array}
$$

Figure 5.5: The residue $\Gamma_2$

$$
\begin{array}{ccc}
(3) & U_4(3) & \\
0 & 1 & B = 3^{1+2}.4 \\
\circ & \circ & \\
3 & 9 & \\
112 & 280 & [3^4]L_2(9) = [3^{1+4}.2]S_4 \\
\end{array}
$$

Figure 5.6: The residue $\Gamma_0$
5.3 Testing the axioms

\[ G = \text{Suz} \]
\[ 448, 345, 497, 600 \]

\[ G_0 = [3]U_4(3).2 \]
\[ 19, 595, 520 \]

\[ G_1 = [3^{3+4}.2.2]S_4.2 \]
\[ 139, 968 \]

\[ G_2 = [3^5]M_{11} \]
\[ 1,924,560 \]

\[ 3^{2+4}.2S_4.2 \]
\[ 69, 984 \]

\[ 3^5L_2(9).2 \]
\[ 174,960 \]

\[ 3^{5+2}.2^{2+2} \]
\[ 34,992 \]

\[ B = 3^{5+2}.2^{1+2} \]
\[ 17, 496 \]

Figure 5.7: The boolean lattice related to the Buekenhout-Ronan rank 3 geometry
$B < G_{01} := [3]3^{1+4}.2S_4.2$ and the natural action of $S_4$ on 4 points is 4-transitive. Since $[G_{01} : B] = 4$, the action is 2-transitive. Secondly $[G_{02} : B] = 10$ where $G_{02} := [3]3^4L_2(9).2$. Once again, the natural action of $L_2(9)$ on 10 points is 2-transitive. Finally $[G_{12} : B] = 2$ where $G_{12} := [3^4].3.3^2.2^{1+2}.2$ and the action is trivially 2-transitive.

We now state a theorem that sums up these results. It is due to Buekenhout, Leemans and the author.

**Theorem 5.3.1.** The rank 3 Buekenhout-Ronan geometry for Suz is a BCRLD-geometry which moreover satisfies (APT).

**Proof.** It only remains to show that $\Gamma$ has apartments. Thanks to the work of Patterson and Wong [48] based on an article of Stellmacher [53], we can draw the point-collinearity graph of the geometry $\Gamma$. This graph is distance regular of diameter 4 and the group $Suz.2$ acts on it (see Figure 5.8). The structure of this graph is also detailed in Brouwer et al. [4].

Fixing some point $p$, we let $a, b, c, d$ denote points of the graph at distance 1, 2, 3 and 4 from $p$ in suborbits $A, B, C$ and $D$ respectively. Denote the points of $X$ that are collinear with $y$ by $X(y)$. We will prove that $A(b) = K_{4,4}$ (a complete bipartite graph on two times 4 points). If we show this, then we can prove easily the existence of apartments in $\Gamma$. Indeed, suppose that $A(b) = K_{4,4}$ and fix $b \in B$ a point at distance 2 from $p$. We can see from the collinearity graph that the points $p$ and $b$ are adjacent to 8 common points in the orbit $A$. It is now clear that there is an octahedron in the graph, as is suggested by Figure 5.9, because those 8 points are a complete bipartite graph on twice 4 points:
there is a square in $A(b)$ and each of its vertices is joined to $p$ and $b$.

This octahedron (or the hemioctahedron) is the thin subgeometry of $\Gamma$ that we are looking for since it is thin, flag-transitive, residually connected and has the same Coxeter diagram as $\Gamma$. This shows that it is enough to prove that the suborbit of $A$ of 8 points that appears when we fix $p$ and $b$ is $K_{4,4}$.

The stabilizer of the point $p$ is $G_p = 3.U_4(3).2$. The subgroup $G_{p,b}$ of $G_p$ is the stabilizer of both $p$ and $b$. From the collinearity map, we see that this group has index 8,505 in $G_p$. We have $|G_p|/8,505 = 2,304$ and $2,304 = 1152 \times 2$. Thanks to the Atlas [25], we deduce that the
\( N = [2](2 \times S_4) \)

\[
\begin{array}{c|c|c}
\Sigma & 0 & 1 & 2 \\
\hline
1 & 1 & 1 \\
6 & 12 & 8 \\
\hline
H & [2]
\end{array}
\]

Figure 5.10: An apartment of the Buekenhout rank 3 geometry

The group acting on the 8 points in \( A \) is the group \( (2(A_4 \times A_4).4).2 \) which is of order 2,304 and which is maximal in \( G_p \). Using the software NAUTY available on McKay’s homepage [45], we can generate the list of vertex-transitive graphs on 8 vertices (this list is already available on Kocay’s homepage [39]). There exist only 10 such graphs. Using the Lagrange Theorem to determine the order of the automorphism group of each of them, we see that the only such graph with automorphism group of order 1,152 is \( K_{4,4} \).

After building the geometry \( \Gamma \) on MAGMA, we let the algorithm HasApartment5 described in Chapter 6 run on it. After one hour, it answered positively and provided the apartment \( \Sigma \) whose diagram is given in Figure 5.10. The group \( N \) is isomorphic to \( [2](2 \times S_4) \). For the record, note that if we run HasApartmentInv instead, we obtain the very same apartment after two minutes of computation.

### 5.3.3 The Buekenhout rank 3 geometry

This geometry first appeared in Buekenhout [7] and was constructed using work about polar spaces in Buekenhout [6] and in Buekenhout and Hubaut [16]. In Figure 5.12 are the maximal parabolic subgroups and the Borel subgroup [5].

We can draw the boolean lattice in Figure 5.11 using those informations and combining them with the ATLAS [25]. Looking at the boolean lattice in Figure 5.11, we see that the geometry \( \Gamma \) is (F), (FT) and
Figure 5.11: The boolean lattice related to the Buekenhout rank 3 geometry
(RC). Since all the inclusions are maximal, $\Gamma$ is (RPRI) and therefore (PRI) and (RW PRI) by virtue of Lemma 2.9.9. This geometry clearly satisfies (IP)$_2$. It is (2T)$_1$ as well. This follows from the fact that on the one hand the indices of the Borel subgroup in the minimal parabolic subgroups $G_{01}$ and $G_{02}$ is 5 and that the action of $A_5$ on five points is 3-transitive, hence 2-transitive; on the other hand, $[G_{12} : B] = 2$ so the action is necessarily 2-transitive. Therefore $\Gamma$ is (2T)$_1$.

After implementing this geometry in Magma [3], we ran the algorithm HasApartmentInv (see Section 6.1.3) on it and we obtained an original result: Magma answered positively. Using the usual notations, there exists an apartment $\Sigma(N, \{N_0, N_1, N_2\})$ with $H := 1$ and $N$ a group of order 192. According to Magma [3], the group $N$ is nonabelian, solvable, not nilpotent and not perfect. The apartment $\Sigma$ belongs to the diagram given in Figure 5.13.

Now we want to get a better understanding of $\Sigma$. Note first that geometries belonging to a linear diagram with gonality $\{3, 6\}$ are studied in depth in Buekenhout [11]. Moreover, the geometry $\Sigma$ is already known as a regular polytope (see Hartley [32]). The Euler characteristic $\chi$ of $\Sigma$ is given by

$$\chi = 16 - 48 + 32 = 0 = 2g - 2 \iff g = 1. \quad (5.3.1)$$

Hence the apartment $\Sigma$ can be viewed as a tiling of a torus with 16 hexagons because a torus is a surface of genus $g = 1$. Hartley [32] states that the polytope $\Sigma$ is toroidal, locally spherical and orientable.

Figure 5.15 provides a realization of $\Sigma$; by identifying opposite edges

\[1\text{ Courtesy of Julien Meyer.}\]
5.3 Testing the axioms

\[ N = 2^3 : S_4 \]

\[
\begin{array}{cccc}
\Sigma & 0 & 1 & 6 & 2 \\
1 & 1 & 1 & 1 \\
16 & 48 & 32 \\
D_{12} & 2^2 & S_3 \\
\end{array}
\]

\[ H = 1 \]

Figure 5.13: The diagram of an apartment \( \Sigma \) of \( \Gamma \)

\[ N = 3^{2+1} : 2^2 \]

\[
\begin{array}{cccc}
\Xi & 0 & 1 & 6 & 2 \\
1 & 1 & 1 & 1 \\
9 & 27 & 18 \\
D_{12} & 2^2 & S_3 \\
\end{array}
\]

\[ H = 1 \]

Figure 5.14: The diagram of an apartment \( \Xi \) of \( \Gamma \)

Figure 5.15: A tesselation of a torus with 16 hexagons
of hexagons that are of the same color, we obtain a torus tiled with 16 hexagons.

We also let the algorithm HasApartment5 run on this geometry and we obtained another type of apartment after about two hours of computation. It is the geometry $\Xi$ that belongs to the diagram given in Figure 5.14. The group $N$ is of order 108. This apartment can also be viewed as a tesselation of a torus with 9 hexagons this time. The geometry $\Xi$ is known as a regular polytope (see Hartley [32]).

This example shows that a coset geometry may possess different types of apartments in our sense. This contrasts with the fact that in a building, all the apartments are isomorphic.

We now state a theorem that sums up those results.

**Theorem 5.3.2.** The Buekenhout rank 3 geometry $\Gamma$ for $Suz$ is a BCDL-geometry that satisfies (APT). Moreover $\Gamma$ possesses at least two different types of apartments in the combinatorial sense whose stabilizer in $Suz$ is flag-transitive.

### 5.3.4 The Ronan-Smith rank 3 geometry

We recall that the rank 3 Ronan-Smith geometry $\Gamma$ for $Suz$ belongs to the diagram given in Figure 5.16. Such a geometry is called a GAB, i.e. a geometry that is almost a building. Note that the type-preserving automorphism group of this geometry is $\text{Aut}(Suz) = Suz.2$.

A geometric construction of the geometry $\Gamma$ is published in Ronan and Smith [50]. This construction involves the Leech lattice $\Lambda$ as well as the Golay code and the Conway group $Co_1$. 
Another geometric construction is provided in Yoshiara [63]. The aim of Yoshiara's construction is to 'provide a non-group-theoretical description of the above GAB associated with $\text{Suz}$, using natural notions with respect to the complex Leech lattice.'

However we do not detail these constructions since it would drive us too far away from the original goal of this dissertation.

We can draw the boolean lattice related to this geometry as in Figure 5.17. The maximal inclusions can be proved combining the Atlass [25] and Magma [3]. This geometry is clearly (F), (FT), (RC) and (IP)$_2$. Since all the inclusions are indeed maximal, the geometry $\Gamma$ is (RPri), thus (Pri) and (RWPri) by virtue of Lemma 2.9.9. The action of the minimal parabolic subgroups on the cosets of the Borel subgroup are 2-transitive using the fact that the action of $S_4$ on three points is 3-transitive, that the action of $A_4$ on three points is 2-transitive and that the action of $A_5$ on five points is 3-transitive. Therefore, $\Gamma$ is a BCDL-geometry.

Since this geometry is very close to a building, it seems natural to guess that it should have apartments. In fact, $\Gamma$ has apartments in the combinatorial sense but does not satisfy (Apt) because the stabilizer $N$ in $\text{Suz}$ of a combinatorial apartment $A$ is not transitive on the set of chambers of $A$.

First we let the algorithms HasApartmentInv and HasApartment run on $\Gamma$ (see Chapter 6 for an implementation of each of them). The algorithm HasApartmentInv finished to run after about three days and answered negatively; after eight weeks of calculations, we stopped the algorithm HasApartment and we did not obtain any result. Naturally, HasApartmentInv does not provide a totally satisfying answer when it answers negatively, by virtue of the reasons detailed in Section 4.3. Therefore we looked for a computer free proof of the fact that $\Gamma$ does not have apartments in our sense.

Let us have a look at the residue $\Gamma_0$ and in particular at $G_0 = [2^{4+6} : 3]Sp_4'(2)$. The residue $\Gamma_0(G_0, \{G_01, G_02\})$ is a rank 2 geometry such that $K := 2^{4+6} : 3$ is a normal subgroup of $G_0$, $G_01$, $G_02$ and $B$. We now invoke Theorem 5.3.3.

**Theorem 5.3.3.** (Tits [57]) If $\Gamma(G, \{G_i \mid i \in I\})$ is a geometry and if $K$ is a normal subgroup of $G$ such that for every $G_i, i \in I, K \leq G_i$, then $\Gamma(G, \{G_i \mid i \in I\}) \cong \Gamma'(G/K, \{G_i/K \mid i \in I\})$.

It is well known that $Sp_4'(2) \cong A_6$. By taking the quotient by
Figure 5.17: The boolean lattice related to the Ronan-Smith rank 3 geometry
5.3 Testing the axioms

Figure 5.18: The symplectic quadrangle with 15 points and 15 lines

\[
\begin{array}{cccc}
\Xi & 0 & 1 & H = D_8 \\
2 & 2 \\
15 & 15 \\
S_4 & S_4
\end{array}
\]

Figure 5.19: The symplectic generalized quadrangle for \( Sp'_4(2) \cong A_6 \)

\( K \), we obtain the geometry \( \Gamma'_0(G_0/K, \{ G_{01}/K, G_{02}/K \} ) \) belonging to the diagram given in Figure 5.19 (see also [15]). It is the generalized quadrangle of Figure 5.18\(^2\) on which we let \( Sp'_4(2) \cong A_6 \) act. Let us call \( \Xi \) this geometry.

The geometry \( \Xi \) has no apartment in our sense; this is known from Buekenhout and Leemans [17]. Indeed, the stabilizer \( N \) of a combinatorial apartment \( \mathcal{A} \) of \( \Xi \) has two orbits on the set of chambers of \( \mathcal{A} \). Therefore, any geometry that has \( \Xi \) as a residue cannot satisfy \( \text{(Apt)} \) by virtue of Theorem 2.10.8. We thus just proved Theorem 5.3.4.

**Theorem 5.3.4.** The Ronan-Smith rank 3 geometry \( \Gamma \) is a BCDL-geometry. It does not satisfy \( \text{(Apt)} \).

Theorem 5.3.4 confirms a result mentioned by Yoshiara [63].

Since we unexpectedly could not find an apartment for \( \Gamma \), we considered this geometry for the group \( \text{Aut}(Suz) = Suz.2 \) which is the

\(^2\)Source: http://symomega.wordpress.com
Geometries for the sporadic simple group of Suzuki

\[ Sp_4(2) \cong S_6 \]

<table>
<thead>
<tr>
<th>( \Theta )</th>
<th>0</th>
<th>1</th>
<th>( H = 2 \times D_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2 ( \times S_4 )</td>
<td>2 ( \times S_4 )</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.20: The symplectic generalized quadrangle for \( Sp_4(2) \cong S_6 \)

type-preserving automorphism group of the geometry, i.e. we considered the geometry \( \Upsilon(Suz.2, \{ \tilde{G}_0, \tilde{G}_1, \tilde{G}_2 \}) \) where \( \tilde{G}_0 \cong [2^{4+6} : 3]Sp_4(2), \)
\( \tilde{G}_1 \cong [2^{2+8}] : (A_5 \times S_3), 2 \) and \( \tilde{G}_2 \cong [2^{1+6}]U_4(2). 2. \) With this geometry,
the arguments we gave for \( \Gamma \) favorising the nonexistence of apartment are not valid anymore. Indeed, let us consider the residue \( \Upsilon_0 \) and in particular the maximal parabolic subgroup \( \tilde{G}_0 \) of \( \Upsilon \); it is now isomorphic to \( [2^{4+6} : 3]S_6. \) Taking the quotient by the kernel \( \tilde{G}_0/(2^{4+6} : 3), \)
we obtain a geometry \( \Theta \) which is the symplectic generalized quadrangle with \( S_6 \cong Sp_4(2) \) acting. Now \( \Theta \) has apartments in our sense.

We first detail the geometry \( \Theta \) before giving a possible strategy using the apartments of \( \Theta \) to build an apartment for \( \Upsilon. \) This strategy may be extended to any geometry. However we do not state that this approach provides a full list of the apartments of a geometry. It deserves closer attention before we can state so.

We can count the number of combinatorial apartments in \( \Theta, \) i.e. the number of quadrangles; a quadrangle is determined by 4 (unordered) points. There are 15 points in the generalized quadrangle; there are 3 points per line. It is thus easy to see that there are

\[
\frac{15 \times 6 \times 4 \times 2}{4 \times 2} = 90 \quad (5.3.2)
\]

quadrangles. The stabilizer in \( S_6 \) of a quadrangle is isomorphic to \( D_8 \) since \( \frac{720}{90} = 8. \) Now, given a chamber \( B, \) it is easy to see that there are

\[
2 \times 2 \times 2 \times 2 = 16 \quad (5.3.3)
\]

apartments \( N \) that contain \( B. \) We thus get a firm grip on the apartments of \( \Theta: \) there are geometries \( \Omega(D_8, \{ C_2, C_2 \}) \), with a trivial Borel subgroup.
5.3 Testing the axioms

At this point, we may envisage the following strategy using these results to build an apartment for \( \Upsilon \). It is based on the fact that, given an apartment \( \Sigma \) of a geometry \( \Gamma \), then any residue of rank at least 2 of \( \Sigma \) is the apartment of a corresponding residue of \( \Gamma \). This is essentially Theorem 2.10.8.

There is a morphism \( \varphi \) from \( \tilde{G}_0 \) to the quotient \( \tilde{G}_0/(2^{4+6} : 3) \) that arises naturally. Since we know the apartments \( \Omega(Q, \{Q_1, Q_2\}) \) of \( \Theta \), we can build an apartment \( \Sigma_0 \) of the residue \( \Upsilon_0 \): it suffices to take \( N_0 \in \varphi^{-1}(Q) \). Now we define \( N_{01} := N_0 \cap G_1 \) and \( N_{02} := N_0 \cap G_2 \). We then set \( \Sigma_0(N_0, \{N_{01}, N_{02}\}) \) and we call the Borel subgroup \( H \).

At this stage, we come back to the ideas we develop in Section 4.3. First we take a generator \( r_2 \) (resp. \( r_1 \)) of \( Q_2 \) (resp. \( Q_1 \)) and we take a representant \( \rho_2 \) (resp. \( \rho_1 \)) of the preimage \( \varphi^{-1}(r_2) \) (resp. \( \varphi^{-1}(r_1) \)). Without loss of generality, we can choose \( \rho_2 \) (resp. \( \rho_1 \)) such that \( N_{01} = \langle H, \rho_2 \rangle \) (resp. \( N_{02} = \langle H, \rho_1 \rangle \)). Next we determine the set \( S_2 \) of elements in \( \tilde{G}_2 \) as in Lemma 4.3.1. Then we try to build an apartment for \( \Upsilon_1 \) (resp. \( \Upsilon_2 \)) with one of its maximal parabolic being \( N_{02} \) (resp. \( N_{01} \)). We do this by looking for an element \( \rho_0 \in S_2 \) such that \( N := \langle H, \rho_0, \rho_1, \rho_2 \rangle \) provides an apartment \( \Sigma(N, \{N_0, \langle H, \rho_0, \rho_2 \rangle, \langle H, \rho_0, \rho_1 \rangle \}) \) of \( \Upsilon \).

We started to apply this method on Magma [3] but we did not manage to obtain a positive answer by the time we finished the redaction of the dissertation. This deserves close attention for further study.

Note that Yoshiara [62] did not study the existence of transitive apartments in the geometry \( \Upsilon \).

5.3.5 The Leemans rank 6 geometry

The rank 6 Leemans geometry \( \Gamma \) belongs to the diagram given in Figure 5.21 (see Leemans [43]).

This geometry belongs to a family of geometries related to the Suzuki tower. We first recall the description of the Suzuki tower given by Tits [58] and then we describe briefly how to build geometrically this geometry.

Let \( I = \{0, 1, \ldots, m\} \), let \( \Delta_i \) denote a graph, let \( X_i \) be its vertex-set, let \( E_i \) be its edge-set, let \( p_i \) denote a point of \( X_i \) and let \( G_i \) be an automorphism group of the graph \( \Delta_i \). A tower is a triple \( (G_i; \Delta_i, p_i) \) where \( i \in I \) such that, for \( j \in I - \{0\} \):

1. \( \text{Aut}(\Delta_i) \) acts transitively on \( X_i \);
2. Let \( Y_{j-1} = \{ x \in X_j \mid \{p_j, x\} \in E_j \} \), we have \( X_{j-1} = X_j - Y_{j-1} - p_j \).

The graph \( \Delta_{j-1} \) is the induced subgraph of \( \Delta_j \) having \( X_{j-1} \) as vertex-set.

3. \( G_i \) is a proper subgroup of \( \text{Aut}(\Delta_i) \), always of index 2. We have \( \text{Aut}(G_i) = \text{Aut}(G'_i) = \text{Aut}(\Delta_i) \). The group \( \tilde{G}_{j-1} = (G_j)_{p_j} \) is transitive on \( Y_{j-1} \) and induces \( G_{j-1} \) on \( X_{j-1} \).

Suzuki [56] constructed a tower of graphs that is today known as the Suzuki tower [58]. This tower gives a sequence of five groups that are

\[
L_3(2) < U_3(3) < J_2 < G_2(4) < \text{Suz}.
\]

We now sketch the construction of the Suzuki tower. This construction is inductive. Start with the incidence graph of the Fano plane and define \( X_0 \) to be its vertex-set, that is, a set of 14 elements consisting in the 7 points and the 7 lines that constitute that plane. The edges of \( \Delta_0 \) are of two types. First, a point (resp. line) is adjacent to all the other points (resp. lines). Second, a point is adjacent to a line if and only if it is on that line. Now we define inductively the sets \( Y_{j-1} \), where \( j \in I \setminus \{0\} \). The set \( Y_{j-1} \) is the set of centres of Sylow 2-subgroups of \( G_{j-1} \). Again we define two types of edges. If \( C, C' \in Y_{j-1} \), where \( C \neq C' \), and if \( x \in X_{j-1} \), then \( \{x, C\} \in E_j \) if and only if the stabilizer in \( C \) of \( x \) is the identity and \( \{C, C'\} \in E_j \) if and only if one of the following two conditions hold: \( C \) and \( C' \) commute (viewed as subgroups of \( G_{j-1} \)), or no element of \( Y_{j-1} \) commutes simultaneously with \( C \) and \( C' \). Table 5.1 gives some parameters of the Suzuki tower.

In [42], Leemans built a rank 4 geometry for the group \( J_2 \) over the diagram given in Figure 5.22 (this geometry is the geometry labeled 4.3

Figure 5.21: The diagram of the Leemans rank 6 geometry for \( \text{Suz} \)
5.3 Testing the axioms

<table>
<thead>
<tr>
<th>$i$</th>
<th>$G_i$</th>
<th>$X_i$</th>
<th>$Y_i$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>$L_3(2) = L_2(7)$</td>
<td>14</td>
<td>21</td>
</tr>
<tr>
<td>1</td>
<td>$U_3(3) = G_2(2)'$</td>
<td>36</td>
<td>63</td>
</tr>
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<td>2</td>
<td>$J_2$</td>
<td>100</td>
<td>315</td>
</tr>
<tr>
<td>3</td>
<td>$G_2(4)$</td>
<td>416</td>
<td>1365</td>
</tr>
<tr>
<td>4</td>
<td>$Suz$</td>
<td>1782</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1: Some parameters of the Suzuki tower

for $J_2$ in [44]). This geometry is BCDL. It has 100 elements of type 1, 100 elements of type 2, 1575 elements of type 3 and 1575 elements of type 4. The stabilizer of an element of type 1 is isomorphic to $U_3(3)$ and the stabilizer of a flag of type $\{1, 2\}$ is $L_3(2)$. Thus there are three geometries in bijection with the three first ‘floors’ of the Suzuki tower. Considering this, it was tempting to try to extend this geometry to a rank 5 geometry for $G_2(4)$ and then again to a rank 6 geometry for $Suz$. Leemans managed to build such geometries and gives a construction in [43].

He proved that the rank 6 geometry he got for $Suz$ is BCDL as well. For a proof of this fact, we refer the interested reader to Theorem 1 in [43]. Unfortunately, this geometry does not satisfy (APT). Indeed, since the Leemans rank 4 geometry for $J_2$ does not satisfy (APT) (see Table 4.3), Theorem 2.10.8 implies that the Leemans rank 6 geometry for $Suz$ does not as well. However, we have to mention that to our knowledge the fact that the Leemans rank 4 geometry for $J_2$ does not satisfy (APT) depends on computer arguments. Therefore it deserves close attention for a further study.

We sum up these results under the following Theorem.

**Theorem 5.3.5.** The Leemans rank 6 geometry for $Suz$ is BCDL. It does not satisfy (APT).
Figure 5.22: The diagram of the Leemans rank 4 geometry for $J_2$
Chapter 6

Appendix

6.1 Magma implementations

In this last chapter, we present the implementations of the algorithms we use in MAGMA [3]. For each of them, we detail the inputs and outputs. These programs are available on request to the author\(^1\).

6.1.1 Algorithms related to gonality

The algorithm Gonality

Here is the implementation of the algorithm described in Chapter 3.

Input:

- a group \( g \);
- a subgroup \( g_0 \) of \( g \);
- a subgroup \( g_1 \) of \( g \).

Output:

- the smallest integer \( s_k \) as in Lemma 3.2.1.

```
function Gonality(g,g0,g1);
    a := g0;
    b := g1;
```

\(^1\)tconnor@ulb.ac.be
ab := a meet b;
o1 := Index(a, ab);
o2 := Index(b, ab);
dp := 1;
p := 1;
s := a;
t := b;
lastp := 1;
dl := 1;
l := 0;
lastl := 0;
gon := 1;
check := true;
first := true;
while #s ne #g do
  s := {x*y : x in s, y in b};
  if first then
    lastl := lastp*o1;
    first := false;
  else
    lastl := lastp*(o1-1);
  end if;
  l := l+lastl;
  if check and #s eq l*#b then
    gon := gon + 1;
  else
    check := false;
    return gon;
  end if;
temp := l;
l := p;
p := temp;
temp := lastl;
lastl := lastp;
lastp := temp;
temp := o1;
o1 := o2;
o2 := temp;
temp := a;
The algorithm CoxeterMatrix and the algorithm CoxeterMatrix2

The following two algorithms have the same input and the same output. They calculate the Coxeter diagram of a coset geometry.

Input:

- a coset geometry CG.

Output:

- the Coxeter matrix of CG.

The difference between them is that CoxeterMatrix uses the already implemented MAGMA function Diagram while CoxeterMatrix2 uses the algorithm Gonality.

function CoxeterMatrix(CG)

d,v,e := Diagram (CG);
MaxPara := MaximalParabolics (CG);
RankCG := Rank(CG);
CoxMat := [[0 : i in [1..RankCG]] : j in [1..RankCG] ];
for i := 1 to RankCG-1 do
    CoxMat[i,i] := 1;
    for j := i+1 to RankCG do
        lab := Label (e!{i,j});
        CoxMat[i,j] := lab[2];
        CoxMat[j,i] := lab[2];
    end for;
end for;
CoxMat[RankCG,RankCG] := 1;
return CoxMat;
end function;
function CoxeterMatrix2(CG)
    G:=Group(CG);
    MaxPara := MaximalParabolics(CG);
    RankCG := Rank(CG);
    CoxMat := [[0 : i in [1..RankCG]] : j in [1..RankCG] ];
    for i := 1 to RankCG-1 do
        CoxMat[i,i] := 2;
        for j := i+1 to RankCG do
            CoxMat[i,j] := Gonality(G,MaxPara[i],MaxPara[j]);
            CoxMat[j,i] := CoxMat[i,j];
        end for;
    end for;
    CoxMat[RankCG,RankCG] := 2;
    return CoxMat;
end function;

6.1.2 The improved Buekenhout-Leemans algorithm for apartments

We implemented different versions of this algorithm. Those versions differ in the inputs and in time execution. For example Apartment4 requires the Coxeter matrix of the coset geometry, while Apartment3 does not. Hence the first one is faster if one already knows the Coxeter matrix. In these implementations, we included conditions of Lemma 4.3.2.

The algorithm Apartment4

Input:

- a coset geometry CG;
- the Coxeter matrix CoxCG of CG;
- the list of the maximal parabolic subgroups MaxPara of CG;
- a boolean StopAfterFound: if it is set to be true, the algorithm stops as soon as one apartment is found; while if it is set to be false, the algorithm looks for all the apartments in the geometry.
6.1 Magma implementations

Output:

- a boolean \( \text{ok} \): it contains \texttt{true} if and only if the computer found an apartment;

- a list \texttt{ListOfN} which contains the list of subgroups \( N \) as in Definition 2.10.5.

```magma
function Apartment4(CG, CoxCG, MaxPara, StopAfterFound)
    ok := false;
    SetOfN := {@@};
    G := Group(CG);
    RankCG := Rank(CG);
    B := Borel(CG);
    lB := SubgroupLattice(B);
    SchlafliSymb := \{2\} join \{CoxCG[i,j] : j in \{i+1..RankCG\}, i in \{1..RankCG-1\}\};
    PS := 2*LCM(SchlafliSymb);
    for i := 1 to #lB do
        Bi := Group(lB!i);
        Q := Normalizer(G, Bi);
        if (#Q mod (PS*#Group(lB!i))) eq 0 then
            lQ := SubgroupLattice(Q);
            for j := 1 to #lQ do
                N1 := Group(lQ!j);
                if (#N1 mod (PS*#Group(lB!i))) eq 0 then
                    NN := Normalizer(G,N1);
                    t := Transversal(G,NN);
                    for k in t do
                        N := N1^k;
                        SCG := CosetGeometry(N,\{@N meet x : x in MaxPara@\});
                        RankSCG := Rank(SCG);
                        if (N meet B eq Bi and RankSCG eq RankCG and
                            IsThin(SCG) and IsFTGeometry(SCG) and IsRC(SCG)) then
                            dd, vv, ee := Diagram(SCG);
                            CoxSCG := CoxeterMatrix(SCG);
                            if CoxCG eq CoxSCG then
                                if (#sub<G|B,N> eq #G) then
                                    ok := true;
                                    SetOfN := SetOfN join {@@};
    ```
The algorithm \texttt{Apartment3}

This algorithm checks if a coset geometry satisfies \texttt{Apt}_r for some $r \in \mathbb{N}$.

Input:
\begin{itemize}
  \item a coset geometry \texttt{CG};
  \item a positive integer \texttt{r};
  \item a boolean \texttt{StopAfterFound}: if it is set to be \texttt{true}, the algorithm stops as soon as one apartment is found; while if it is set to be \texttt{false}, the algorithm looks for all the apartments in the geometry.
\end{itemize}

Output:
\begin{itemize}
  \item \texttt{true} if and only if the computer found an apartment;
  \item if \texttt{r} is equal to the rank of the coset geometry, it returns also a list \texttt{List0fN} which contains the list of subgroups $N$ as in Definition 2.10.5.
\end{itemize}

function \texttt{Apartment3(CG,r,StopAfterFound)}
\begin{verbatim}
G := Group(CG);
RankCG := Rank(CG);
if r eq RankCG then
  MaxPara := MaximalParabolics(CG);
  CoxCG := CoxeterMatrix(CG);
end if;
end if;
end if;
end for;
end for;
end for;
return ok, Set0fN;
end function;
\end{verbatim
return Apartment4(CG,CoxCG,MaxPara,StopAfterFound);
else
  t := Types(CG);
  for x in Subsets(Set(t),#t-r) do
    Res := Residue(CG,x);
    MaxPara := MaximalParabolics(Res);
    CoxRes := CoxeterMatrix(Res);
    HasAparts := Apartment4(Res,CoxRes,MaxPara,true);
    if not HasAparts then
      return false,{@@};
    end if;
  end for;
  return true,{@@};
end if;
end function;

The algorithm Apartment

This is the most portable version of this algorithm. It runs Apartment3 with the instance (CG,Rank(CG),true) where CG is a coset geometry.

function Apartment(CG)
  return Apartment3(CG,Rank(CG),true);
end function;

The algorithm ApartmentRes

This version of the Buekenhout-Leemans algorithm makes use of Theorem 2.10.8. It may be useful on high rank geometries since it determines if the residues of a geometry, listed in increasing rank, satisfy or not APT. As soon as one residue does not, the algorithm answers negatively. Clearly, this may be a good strategy only if one suspects that the geometry has no apartment.

Input:

- a coset geometry CG.

Output:

- false as soon as one residue that has no apartment is found, true if the geometry has apartments.
function ApartmentRes(CG)
  t:=Types(CG);
  r:=#t;
  for k:=2-r to 0 do
    for x in Subsets(Set(t),-k) do
      if (not Apartment3(Residue(CG,x),r+k,true)) then
        return false;
      end if;
    end for;
  end for;
  return true;
end function;

6.1.3 The new algorithm HasApartment and variations

The algorithm Conditions

This function is part of the algorithm HasApartment. It consists in checking if conditions as they are stated after Lemma 4.3.1 are satisfied.

Input:

- \( x \) an element of a minimal parabolic;
- \( \text{Typex} \) an index determining the minimal parabolic subgroup to which \( x \) belongs;
- \( \text{SchlaflIqSym} \) the Coxeter matrix of the geometry;
- \( \rho \) a vector built using a backtracking method which contains the elements of the minimal parabolic subgroups as detailed after Lemma 4.3.1;
- \( G \) the group of the coset geometry;
- \( H \) the potential Borel subgroup of the apartment as detailed after Lemma 4.3.1.

Output:

- \text{true} if \( x \) satisfies the conditions detailed after Lemma 4.3.1, \text{false} otherwise.
function Conditions(x,Typex,SchlafliSymb,rho,G,H)
for i:=1 to Typex-1 do
  y := rho[i];
  if not (#sub<G|H,x,y> eq #H*2*SchlafliSymb[i,Typex]) then
    return false;
  end if;
end for;
return true;
end function;

The algorithm GoodElts

This algorithm is the part of the algorithm HasApartment that selects the vectors of elements as exposed after Lemma 4.3.1.

Input:

- **counter** an index;
- **SchlafliSymb** the Coxeter matrix of the geometry;
- **elements** a vector of sets. The sets contain the elements as in Lemma 4.3.1;
- **rho** a vector built using a backtracking method which contains the elements of the minimal parabolic subgroups as detailed after Lemma 4.3.1;
- **G** the group of the coset geometry;
- **H** the potential Borel subgroup of the apartment as detailed after Lemma 4.3.1;
- **IndexMax** the rank of the geometry;
- **ListOfGoodElts** a vector;
- **ThereAreGoodElts** a boolean intialized at false which becomes true as soon as one good vector rho is built;
Output:

- GoodElts builds a vector ListOfGoodElts of all possible combinations of elements of the minimal parabolic subgroups as detailed after Lemma 4.3.1.

```plaintext
procedure GoodElts(counter,conditions,~elements,~rho,~G,~H, IndexMax,~ListOfGoodElts,~ThereAreGoodElts)
for x in elements[counter] do
  if Conditions(x,counter,conditions,rho,G,H) then
    rho[counter] := x;
    if counter lt IndexMax then
      GoodElts(counter+1,conditions,~elements,~rho,~G,~H, IndexMax,~ListOfGoodElts,~ThereAreGoodElts);
    else
      elt := rho;
      Append(~ListOfGoodElts,elt);
      ThereAreGoodElts := true;
    end if;
  end if;
end for;
end procedure;
```

The algorithm HasApartmentInv

In this version of our algorithm, we make use of Remark 4.3.3. That is, instead of looking for all elements like it is explained after Lemma 4.3.1, we rather look for involutions in the minimal parabolics that are not in the Borel subgroup. When no such involution is found, we consider the elements as in Lemma 4.3.1. The rest of the algorithm is identical to the algorithm detailed in Section 4.3.

This algorithm is very fast but can miss some apartments since it does not consider all the possible \(n\)-uples of elements as exposed after Lemma 4.3.1. However, practice shows that it may be a good idea first to run this algorithm (unless one wants the whole list of apartments); if it answers positively, then it is totally reliable and for sure faster than any other algorithm we wrote, but if it answers negatively then one has to try another algorithm. Let us recall that on the one hand this algorithm mistook only three times on the 420 geometries available in Leemans [44] and on the other hand it is so fast compared to the other
algorithms that if it erroneously answers negatively, then the time spent
is negligible compared to the time any other algorithm will take as is
shown in Tables 5.3.5.

Input:
- a coset geometry $CG$;
- the Coxeter matrix $CoxCG$ of $CG$;
- the list of the maximal parabolic subgroups $MaxPara$ of $CG$;
- the list of the minimal parabolic subgroups $MinPara$ of $CG$;
- a boolean $StopAfterFound$: if it is set to be $true$, the algorithm
  stops as soon as one apartment is found; while if it is set to be
  $false$, the algorithm looks for all the apartments in the geometry.

Output:
- a boolean $HasAparts$: it contains $true$ if and only if the computer
  found an apartment;
- a list $Classes$ of indices in the subgroup lattice of the Borel in
  which a good subgroup $H$ has been found;
- a list $ListOfN$ which contains the list of subgroups $N$ as in Defi-
  nition 2.10.5.

function HasApartmentInv($CG$, $CoxCG$, $MaxPara$, $MinPara$,
  $StopAfterFound$)
  HasAparts := false;
  Classes := {@@};
  ListOfN := {@@};
  $G$ := Group($CG$);
  $B$ := Borel($CG$);
  $lB$ := SubgroupLattice($B$);
  $RankCG$ := Rank($CG$);
  SchlafliSymb := {2};
  for $i$ := 1 to $RankCG$-1 do
    for $j$ := $i$+1 to $RankCG$ do
      SchlafliSymb := SchlafliSymb join {CoxCG[$i$,$j$]};
end for;
end for;
ProductSymb := 2*LCM(SchlafliSymb);
for i := 1 to #lB do
Rep := Group(1B!i);
if (#Normalizer(G,Rep) mod (#Rep*ProductSymb) eq 0) then
for g in Transversal(B,Normalizer(B,Rep)) do
Candidates := [];
H := Rep^g;
j := 1;
CanContinue := true;
while (CanContinue and (j le RankCG)) do
for k := 1 to RankCG do
Invocations := {y : y in MinPara[k] | not y in B
and Order(y) eq 2};
if #Invocations eq 0 then
q, alpha := quo<Normalizer(MinPara[k],H)|H>;
Invocations := {h*x : h in H, x in {y@@alpha :
y in q | Order(y) eq 2}};
end if;
Append(~Candidates, Invocations);
end for;
if #Candidates[j] eq 0 then
CanContinue := false;
end if;
j := j+1;
end while;
if CanContinue then
rho := [Id(G) : j in [1..RankCG]];
ListOfGoodElts := [];
ThereAreGoodElts := false;
for x in Candidates[1] do
rho[1] := x;
GoodElts(2,CoxCG,^Candidates,^rho,^G,^H,RankCG,
^ListOfGoodElts,^ThereAreGoodElts);
for elt in ListOfGoodElts do
rho := SequenceToSet(elt);
N := sub<G|H,rho>;
BN := sub<G|B,N>;}
if #BN eq #G then
    SubgroupsOfN := {};
    for l := 1 to RankCG do
        SubgroupsOfN := SubgroupsOfN join {N meet MaxPara[l]};
    end for;
    SCG := CosetGeometry(N, SubgroupsOfN);
    if (IsThin(SCG) and IsFTGeometry(SCG) and
        IsResiduallyConnected(SCG)) then
        HasAparts := true;
        Classes := Classes join {@i@};
        ListOfN := ListOfN join {@N@};
        if StopAfterFound then
            return HasAparts, Classes, ListOfN;
        end if;
    end if;
end if;
end for;
end function;

The algorithm HasApartment5

This implementation of the algorithm HasApartment is the analogue of Apartment4. It is the version we used for the time comparisons of Table 5.3.5.

Input:

- a coset geometry CG;
- the Coxeter matrix CoxCG of CG;
- the list of the maximal parabolic subgroups MaxPara of CG;
• the list of the minimal parabolic subgroups $\text{MinPara}$ of $\text{CG}$;

• a boolean $\text{StopAfterFound}$: if it is set to be $\text{true}$, the algorithm stops as soon as one apartment is found; while if it is set to be $\text{false}$, the algorithm looks for all the apartments in the geometry.

Output:

• a boolean $\text{HasAparts}$: it contains $\text{true}$ if and only if the computer found an apartment;

• a list $\text{Classes}$ of indices in the subgroup lattice of the Borel in which a good subgroup $H$ has been found;

• a list $\text{ListOfN}$ which contains the list of subgroups $N$ as in Definition 2.10.5.

function $\text{HasApartment5}(\text{CG},\text{CoxCG},\text{MaxPara},\text{MinPara},$

$\text{StopAfterFound})$

$\text{HasAparts} := \text{false};$

$\text{Classes} := \{\emptyset\};$

$\text{ListOfN} := \{\emptyset\};$

$G := \text{Group}(\text{CG});$

$B := \text{Borel}(\text{CG});$

$lB := \text{SubgroupLattice}(B);$

$\text{RankCG} := \text{Rank}(\text{CG});$

$\text{SchlafliSymb} := \{2\};$

for $i := 1$ to $\text{RankCG}-1$ do

$\text{for } j := i+1$ to $\text{RankCG}$ do

$\text{SchlafliSymb} := \text{SchlafliSymb join } \{\text{CoxCG}[i,j]\};$

end for;
end for;

$\text{ProductSymb} := 2*\text{LCM}(\text{SchlafliSymb});$

for $i := 1$ to $\#lB$ do

$\text{Rep} := \text{Group}(lB[i]);$

if $(\#\text{Normalizer}(G,\text{Rep}) \mod (\#\text{Rep}*\text{ProductSymb}) \text{ eq } 0)$ then

$\text{for } g \text{ in Transversal}(B,\text{Normalizer}(B,\text{Rep}))$ do

$\text{Candidates} := [];$

$H := \text{Rep}^g;$

$j := 1;$

$\text{CanContinue} := \text{true};$

end for;
end if;
end for;
while (CanContinue and (j le RankCG)) do
  for k := 1 to RankCG do
    q, alpha := quo<Normalizer(MinPara[k], H)|H>;
    Append(~Candidates, {G|h*x : h in H, x in {y@alpha :
      y in q | Order(y) eq 2}});
  end for;
  if #Candidates[j] eq 0 then
    CanContinue := false;
  end if;
  j := j+1;
end while;
if CanContinue then
  rho := [Id(G) : j in [1..RankCG]];
  ListOfGoodElts := [];
  ThereAreGoodElts := false;
  for x in Candidates[1] do
    rho[1] := x;
    GoodElts(2, CoxCG, ~Candidates, ~rho, ~G, ~H, RankCG,
      ~ListOfGoodElts, ~ThereAreGoodElts);
    for elt in ListOfGoodElts do
      rho := SequenceToSet(elt);
      N := sub<G|H, rho>;
      BN := sub<G|B, N>;
      if #BN eq #G then
        SubgroupsOfN := {};
        for l := 1 to RankCG do
          SubgroupsOfN := SubgroupsOfN join {N meet
            MaxPara[l]};
        end for;
        SCG := CosetGeometry(N, SubgroupsOfN);
        if (IsThin(SCG) and IsFTGeometry(SCG) and
          IsResiduallyConnected(SCG)) then
          HasAparts := true;
          Classes := Classes join {@i@};
          ListOfN := ListOfN join {N@};
        end if;
        if StopAfterFound then
          return HasAparts, Classes, ListOfN;
        end if;
      end if;
    end for;
  end for;
end if;
The algorithm \( \text{HasApartment3} \)

This implementation of the algorithm \( \text{HasApartment} \) is the analogue of \( \text{Apartment4} \). It checks the property \( \text{Apt}_r \) on a geometry for some positive integer \( r \).

Input:

- a coset geometry \( \text{CG} \);
- a positive integer \( r \);
- a boolean \( \text{StopAfterFound} \): if it is set to be \text{true}, the algorithm stops as soon as one apartment is found; while if it is set to be \text{false}, the algorithm looks for all the apartments in the geometry.

Output:

- \text{true} if and only if the computer found an apartment;
- if \( r \) is equal to the rank of the coset geometry, it returns also a list \( \text{ListOfN} \) which contains the list of subgroups \( N \) as in Definition 2.10.5 and a list \( \text{Classes} \) of indices in the subgroup lattice of the Borel subgroup in which a good subgroup \( H \) has been found.

function \( \text{HasApartment3}(\text{CG}, r, \text{StopAfterFound}) \)

\[
\begin{align*}
\text{G} &:= \text{Group}(\text{CG}); \\
\text{RankCG} &:= \text{Rank}(\text{CG}); \\
\text{if } r \text{ eq } \text{RankCG} \text{ then} \\
\text{if } \text{StopAfterFound} \text{ then}
\end{align*}
\]
MaxPara := MaximalParabolics(CG);
MinPara := MinimalParabolics(CG);
CoxCG := CoxeterMatrix(CG);
HasAparts,Classes,ListOfN := HasApartmentInv(CG,CoxCG,
   MaxPara,MinPara,true);
if not HasAparts then
   HasAparts,Classes,ListOfN := HasApartment5(CG,CoxCG,
   MaxPara,MinPara,true);
end if;
return HasAparts,Classes,ListOfN;
else
   MaxPara := MaximalParabolics(CG);
   MinPara := MinimalParabolics(CG);
   CoxCG := CoxeterMatrix(CG);
   return HasApartment5(CG,CoxCG,MaxPara,MinPara,false);
end if;
else
   t := Types(CG);
   for x in Subsets(Set(t),#t-r) do
      Res := Residue(CG,x);
      MaxPara := MaximalParabolics(Res);
      MinPara := MinimalParabolics(Res);
      CoxRes := CoxeterMatrix(Res);
      if not HasApartment5(Res,CoxRes,MaxPara,MinPara,true)
         then
            return false,@@,@@;
         end if;
   end for;
   return true,@@,@@;
end if;
end function;

The algorithm HasApartment

This is the most portable version. It runs HasApartment3 on the instance (CG,Rank(CG),true).

function HasApartment(CG)
   return HasApartment3(CG,Rank(CG),true);
end function;
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