

An analytical construction of the SRB measures for Baker-type maps

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For a class of dynamical systems, called the axiom-A systems, Sinai, Ruelle and Bowen showed the existence of an invariant measure (SRB measure) weakly attracting the temporal average of any initial distribution that is absolutely continuous with respect to the Lebesgue measure. Recently, the SRB measures were found to be related to the nonequilibrium stationary state distribution functions for thermostated or open systems. In spite of the importance of these SRB measures, it is difficult to handle them analytically because they are often singular functions. In this article, for three kinds of Baker-type maps, the SRB measures are analytically constructed with the aid of a functional equation, which was proposed by de Rham in order to deal with a class of singular functions. We first briefly review the properties of singular functions including those of de Rham. Then, the Baker-type maps are described, one of which is nonconservative but time reversible, the second has a Cantor-like invariant set, and the third is a model of a simple chemical reaction $R \leftrightarrow I \leftrightarrow P$. For the second example, the cases with and without escape are considered. For the last example, we consider the reaction processes in a closed system and in an open system under a flux boundary condition. In all cases, we show that the evolution equation of the distribution functions partially integrated over the unstable direction is very similar to de Rham's functional equation and, employing this analogy, we explicitly construct the SRB measures. © 1998 American Institute of Physics. [S1054-1500(98)02702-5]

Characterization of nonequilibrium stationary states in terms of dynamical ensembles is one of the main questions in statistical mechanics. Recently, the so-called Sinai-Ruelle-Bowen (SRB) measures, which had been studied in dynamical systems theory, were found to be related to nonequilibrium stationary ensembles for thermostated or open systems. The SRB measures fully describe transport properties of the corresponding nonequilibrium stationary states. Also they would provide an important insight about the emergence of irreversibility in reversible dynamical systems, since they do not have time-reversal invariance even when the dynamics is reversible. It is therefore illustrative to know exact forms of the SRB measures, but it is difficult to handle them exactly because they are often singular functions. In this paper, we study three examples of Baker-type maps, which illustrate some aspects of the thermostated and/or open systems: One is nonconservative but time reversible, the second has a Cantor-like invariant set, and the third is a model of a simple chemical reaction such as an isomerization $R \leftrightarrow I \leftrightarrow P$. For those maps, we analytically construct SRB measures with the aid of a new method, where the weak convergence of measures is converted to the strong convergence of partially integrated distribution functions (PIDFs) and the evolution equations for the PIDFs are solved employing the analogy between them and de Rham's functional equations.

I. STATISTICAL MECHANICS AND SRB MEASURES

One of the main questions in statistical mechanics is to characterize nonequilibrium stationary states in terms of dynamical ensembles (cf., e.g., Refs. 1–3). Recently, for thermostated or open systems, stationary nonequilibrium ensembles were found to be related to the so-called Sinai-Ruelle-Bowen (SRB) measures,^{4–15} which had been investigated in dynamical systems theory.^{16–19} In the thermostated systems,^{4–12} a fictitious damping force mimicking a heat reservoir is introduced to avoid an uncontrolled growth of the kinetic energy due to an external driving force. The damping force is chosen so as to make the dynamics dissipative while it preserves time-reversibility. As a result of the dissipation, there exists an attractor of information dimension less than the dimension of phase space and the nonequilibrium stationary state is described by an asymptotic measure, which is an SRB measure. The SRB measure fully characterizes the transport properties, such as the transport law, transport coefficients and their fluctuations. For example, for the driven thermostated Lorentz gas,⁶ the conductivity tensor was calculated, and Ohm's law and Einstein's relation were verified by comparing the averaged current with respect to the SRB measure to the external electric field. On the other hand, open chaotic Hamiltonian systems with a flux boundary condition^{13–15} admit a nonequilibrium stationary state obeying Fick's law that is described by a kind of SRB measure. This measure again characterizes the transport properties. Even for an open Hamiltonian system with an

absorbing boundary condition, an SRB-like measure may describe the long-time behavior of averaged dynamical functions. Such a system has a fractal repeller, that controls the chaotic scattering.^{12,20-26} The unstable manifold of the fractal repeller supports a conditionally invariant measure, which provides the long time limits of averaged dynamical functions^{23,24} and may be of SRB-type (cf. Sec. IV B). We note that the interrelation between the thermostated systems approach and the open systems approach has been discussed by Breymann, Tél and Vollmer.²⁹ In this article, we present an analytical construction of SRB measures for three examples of Baker-type maps, which illustrate some aspects of the thermostated and/or open systems mentioned above. Now we start with the general arguments on the SRB measure.

The long-term behavior of a dynamical system is characterized by an invariant measure μ on an invariant set A , which describes how frequently various parts of A are visited by a given orbit $x(t)$ (with t the time). The invariant measure is said to be ergodic if it cannot be decomposed into different invariant measures. Such an ergodic invariant measure μ satisfies the ergodic theorem.^{1-3,16} In the case of a map S , it asserts that, for any continuous function $\varphi(x)$, we have

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=0}^T \varphi(S^t x) = \frac{\int_A \varphi(x') \mu(dx')}{\int_A \mu(dx')} \quad (x \in A \setminus E \text{ with } \mu(E) = 0). \quad (1)$$

A dynamical system typically admits uncountably many distinct ergodic measures and not all of them are physically observable. One criterion of choosing a physical measure μ is that μ describes the time averages of observables on motions with initial data x randomly sampled with respect to the Lebesgue measure μ_0 .^{16,17}

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=0}^T \varphi(S^t x) = \frac{\int_A \varphi(x') \mu(dx')}{\int_A \mu(dx')} \quad \left(\begin{array}{l} x \in \Sigma \setminus E \text{ with } \Sigma \supset A, \\ \mu_0(\Sigma) > 0 \text{ and } \mu_0(E) = 0 \end{array} \right). \quad (2)$$

Sinai, Ruelle and Bowen showed that a class of dynamical systems, called axiom-A systems, uniquely admit such a physical invariant measure (SRB measure).¹⁶⁻¹⁸ Thus an SRB measure is one for which the ergodic theorem is true for almost every point, x , with respect to the Lebesgue measure μ_0 .

Axiom-A systems are characterized by the hyperbolic structure, i.e., the existence of exponentially stable and exponentially unstable directions which intersect transversally with each other. In case of bijective differentiable maps (i.e., diffeomorphisms), the hyperbolic structure is defined as follows: Consider a given orbit $S^t x$ ($t=0, \pm 1, \pm 2, \dots$) and small deviations δx of the initial value x . Note that there are m independent possible directions of δx when the phase space dimension is m . Assume that the deviations along m_s directions decrease more rapidly than an exponential function $e^{-\lambda t}$ ($\lambda > 0$ and $t \geq 0$) and that the deviations along the

other $m - m_s$ directions grow more rapidly than an exponential function $e^{\lambda t}$ ($t \geq 0$ and the same λ). An invariant set Λ is said to be hyperbolic when (1) for any $x \in \Lambda$, the orbit $S^t x$ has the m_s stable and $m - m_s$ unstable directions as explained above, (2) the stable and unstable directions depend continuously on x and (3) the stable and unstable directions for the point x are mapped by S^t to the corresponding directions for the point $S^t x$.

A point x is said to be nonwandering if the orbit $S^t x$ returns indefinitely often to any neighborhood of its initial point x . If the set Ω of all nonwandering points is hyperbolic and the set of periodic points is dense in Ω , S is called an axiom-A diffeomorphism. In particular, if the whole phase space M is hyperbolic, S is called an Anosov diffeomorphism. The Arnold cat map is an example of an Anosov diffeomorphism.

Invariant measures that are smooth along the unstable directions are called SRB measures. Sinai, Ruelle and Bowen showed that, for axiom-A diffeomorphisms, the SRB measure is the unique physical measure μ describing the time averages (2) of observables of motion with initial data x taken at random with respect to the Lebesgue measure μ_0 .¹⁶⁻¹⁸ For more details on axiom-A systems and SRB measures, see Refs. 16, 17, and 19.

In Gibbs' picture of statistical mechanics, a macroscopic state for an isolated system is described by a phase-space distribution function, and a macroscopic observable by an averaged dynamical function with respect to the distribution. Suppose that the dynamics satisfies the mixing condition with respect to the microcanonical distribution μ_{mc}

$$\lim_{t \rightarrow +\infty} \int_M \varphi(S^t x) \rho_0(x) \mu_{mc}(dx) = \frac{\int_M \varphi(x) \mu_{mc}(dx)}{\int_M \mu_{mc}(dx)}, \quad (3)$$

where M stands for the whole phase space, $\varphi(x)$ is a continuous dynamical function and $\rho_0(x)$ is a normalized initial distribution function. Then, the system exhibits time evolution as expected from statistical thermodynamics. Namely, the distribution function weakly approaches an equilibrium microcanonical distribution and the averaged dynamical functions approach well defined equilibrium values.¹ From this point of view, instead of Eq. (2), it is enough to consider the following condition as a criterion of choosing a physical measure μ :

$$\lim_{t \rightarrow +\infty} \int_M \varphi(S^t x) \rho_0(x) \mu_0(dx) = \frac{\int_A \varphi(x) \mu(dx)}{\int_A \mu(dx)}, \quad (4)$$

where μ_0 stands for the Lebesgue measure, M is the whole phase space, A the attractor and $\varphi(x)$ and $\rho_0(x)$ are, respectively, a continuous dynamical function and a normalized initial distribution function. Sinai, Ruelle and Bowen¹⁸ showed that the SRB measures for the axiom-A systems satisfy Eq. (4) as well. Hence, the measure satisfying Eq. (4) will be referred to as a physical measure. Since the left hand side of Eq. (4) is the average $\langle \varphi \rangle_t$ of φ at time t , Eq. (4) can be generalized to define a physical measure μ for systems with escape^{23,24}

$$\begin{aligned} \lim_{t \rightarrow +\infty} \langle \varphi \rangle_t &= \lim_{t \rightarrow +\infty} \frac{\int_M \varphi(S^t x) \rho_0(x) \mu_0(dx)}{\int_M \chi_M(S^t x) \rho_0(x) \mu_0(dx)} \\ &= \frac{\int_{A'} \varphi(x) \mu(dx)}{\int_{A'} \mu(dx)}, \end{aligned} \tag{5}$$

where χ_M stands for the characteristic function of the phase space M and A' is the support of μ . The denominator of the middle expression is necessary as the total probability is not preserved. When there is no escape, the physical measure defined by Eq. (4) is supported by the attractor and, thus, is invariant. On the other hand, when there is escape, the physical measure defined by Eq. (5) is, in general, not an invariant measure. However, since the ratio $\int_{A'} \varphi(x) \mu(dx) / \int_{A'} \mu(dx)$ does not evolve in time, such a measure is called conditionally invariant.^{23,24} We remark that, when they are of axiom-A type, the systems with escape also possess ‘‘natural’’ invariant measures supported by the fractal repeller, which are specified, e.g., by a variational principle.^{1,12,22–28} It is the natural invariant measure that characterizes the ergodic properties such as the Lyapunov exponents, but not the conditionally invariant physical measure defined by Eq. (5).

Now we revisit thermostated and open systems. As explained before, a thermostated system is dissipative because of the fictitious damping force. Then, a nonequilibrium stationary state is described by an asymptotic SRB measure defined by Eq. (2).^{4–12} For open chaotic Hamiltonian systems with a flux boundary condition, a nonequilibrium stationary state obeying Fick’s law is described by a measure with a fractal structure along the contracting direction.^{13–15} Since the measure is smooth along the expanding direction and can be defined by Eq. (4), it is an SRB measure.¹³ In both cases, those SRB measures fully characterize the transport properties. However, it should be noticed that the two cases are different because the invariant set of an open system is a fractal repeller which is fractal in both the stable and unstable directions and, thus, does not support an SRB measure, while the invariant set of a thermostated system is an attractor which does support an SRB measure.

By applying Eq. (4) for an initial constant density, one obtains a method of constructing an SRB measure for a map:³⁰ (1) Approximate the measure by iterating an initial constant density finite times; (2) calculate the average with its result and (3) take the limit of infinite iterations. Several methods are also proposed where unstable periodic orbits or trajectory segments are used to write down an SRB measure and averages with respect to it (cf. Refs. 5, 7, 8, 10, 21, 22, and 31 and references therein). However, it is not easy to evaluate the convergence rate of the limits in Eqs. (4) and (5), particularly for nonexpanding maps, and to extract exponentially decaying terms from an averaged dynamical function $\langle \varphi \rangle_t$, which are the Pollicott-Ruelle resonances.^{32–34,14,35–38} One of the reasons is that the long-time limit of the measure can be defined only via the ensemble average as shown in Eqs. (4) and (5). For nonexpanding maps, the distribution function itself does not have a well-defined long-time limit. In other words, an asymptotic SRB measure is the *weak* limit of an initial measure. In an analytical construction of SRB measures, which we shall ex-

plain for three Baker-type maps, the weak convergence of measures is converted to the usual convergence (technically speaking, the *strong* convergence) of partially integrated distribution functions. Contrary to the evolution equation of a distribution function (the Frobenius-Perron equation), the evolution equation of a partially integrated distribution is contractive and possesses a well-defined long-time limit. This equation is similar to de Rham’s functional equation,³⁹ which was introduced to deal with singular functions such as continuous functions with zero derivatives almost everywhere, and its contraction rate gives the convergence rate of the averaged dynamical function $\langle \varphi \rangle_t$. In Sec. II, we review some properties of singular functions including those of de Rham’s functional equation. In Sec. III, an SRB measure is constructed for a nonconservative reversible Baker map with the aid of de Rham’s equation. The model illustrates the two fundamental features of the thermostated systems, namely, the phase space contraction and time reversibility, and we discuss the interrelation between the two features. In Sec. IV, a Baker map with a Cantor-like invariant set is studied. When there is no escape, the map possesses a strange attractor, which is a direct product of the unit interval and a Cantor set. On the other hand, when there is escape, the map has an invariant set, which is the direct product of two Cantor sets, and is a simple example of an open system with an absorbing boundary condition. Physical measures for the map with and without escape are constructed with the aid of de Rham’s equation and the natural invariant measure for the map with escape is derived. In Sec. V, we investigate the properties of a Baker map with a flux boundary condition, which mimics a chemical-reaction dynamics with a flux boundary condition (cf. Ref. 40). We show that an SRB-type stationary distribution describes the reaction dynamics and that the slowest relaxation to it is characterized by a decay mode (i.e., the Pollicott-Ruelle resonance), which is a conditionally invariant measure. Section VI is devoted to concluding remarks. Technical details of the construction of measures are presented in Appendixes.

II. SINGULAR FUNCTIONS AND de RHAM EQUATION

Basic tools of our analytical construction of SRB measures are singular functions and de Rham’s functional equation. Singular functions such as the Weierstrass function or the Takagi function were originally introduced as pathological counter examples to the intuitive picture of functions. These singular functions play an important role in chaotic dynamics. A step towards the analytical treatment of singular functions was given by de Rham,³⁹ who showed many of them can be characterized as a unique fixed point of a contraction mapping in a space of functions. In this section, we briefly review the properties of some singular functions and the relation between them and chaotic dynamical systems (cf. Refs. 47 and 51).

The first example of a nowhere differentiable continuous function was given by Weierstrass in 1872⁴¹ (Fig. 1):

$$W_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \tag{6}$$

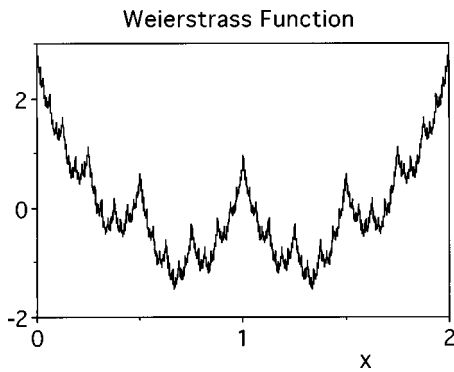


FIG. 1. Weierstrass function for $a=2/3$ and $b=2$.

where $0 < a < 1$, b is a positive number and $ab \geq 1$. Moser⁴² used the Weierstrass function to construct a nowhere differentiable torus for an analytic Anosov system. Yamaguti and Hata⁴³ used it as a generating function of orbits for the logistic map and discussed some generalizations. Also Weierstrass functions are eigenfunctions of the Frobenius-Perron operator for the Renyi map $Sx = rx \pmod{1}$ with r a positive integer.^{35,44}

In 1903, Takagi gave a simpler example of a nowhere differentiable continuous function⁴⁵ (Fig. 2):

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \psi(2^n x), \tag{7}$$

where $\psi(x) = |x - [x + 1/2]|$ and $[y]$ stands for the maximum integer which does not exceed y . In 1930, van der Waerden gave a similar function which is obtained from Eq. (7) by replacing 2^n to 10^n .⁴⁶ In 1957, the Takagi function was rediscovered by de Rham.³⁹ Some generalizations of the Takagi function were discussed by Hata and Yamaguti.^{43,47} The function $T(x) - 1/2$ is the eigenfunction of the Frobenius-Perron operator for the map $Sx = 2x \pmod{1}$ with eigenvalue $1/2$. Also, the Takagi function and related functions were found to describe the nonequilibrium stationary state obeying Fick's law for the multi-Baker map.¹³

In the theory of the Lebesgue integral, there appear non-constant functions with zero derivatives almost everywhere, which are sometimes referred to as Lebesgue's singular

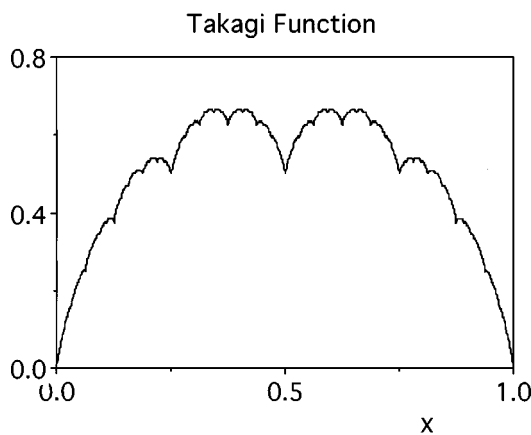


FIG. 2. Takagi function.

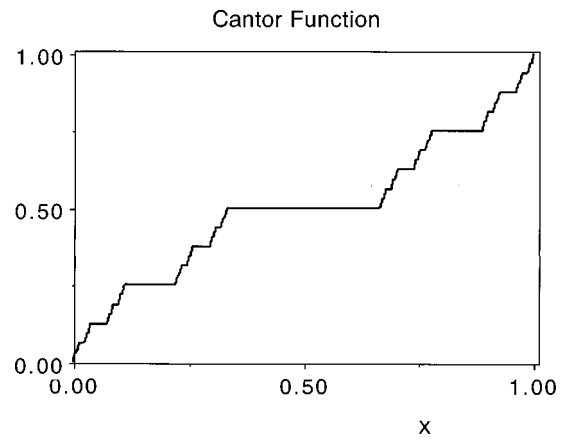


FIG. 3. Cantor function.

functions.^{48,49} One typical example is the Cantor function (Fig. 3). A more interesting example $f_\alpha(x)$ ($0 < \alpha < 1, \alpha \neq 1/2$) is the unique function satisfying

$$f_\alpha(x) = \begin{cases} \alpha f_\alpha(2x), & x \in [0, 1/2] \\ (1 - \alpha)f_\alpha(2x - 1) + \alpha, & x \in [1/2, 1], \end{cases} \tag{8}$$

which is strictly increasing and continuous, but has zero derivatives almost everywhere with respect to the Lebesgue measure⁴⁹ (Fig. 4). Note that such functions do not satisfy the fundamental theorem of calculus⁴⁸

$$f_\alpha(1) - f_\alpha(0) = 1 \neq 0 = \int_0^1 f'_\alpha(x) dx.$$

The function $f_\alpha(x)$ with real α represents a cumulative distribution function of an ergodic measure for the dyadic map $Sx = 2x \pmod{1}$ (cf. examples given on p. 626 of Ref. 16 and on p. 36 of Ref. 50). The eigenfunctionals of the Frobenius-Perron operator for the multi-Bernoulli map and the multi-Baker map can be represented as the Riemann-Stieltjes integrals with respect to $f_\alpha(x)$ with complex α .^{36,37} In Ref. 38, it was shown that, for a class of piecewise linear maps, the left eigenfunctionals of the Frobenius-Perron operator admit a representation in terms of singular functions similar to $f_\alpha(x)$. As pointed out by Hata and Yamaguti,^{43,47} $f_\alpha(x)$ is analytic with respect to the parameter α though it is

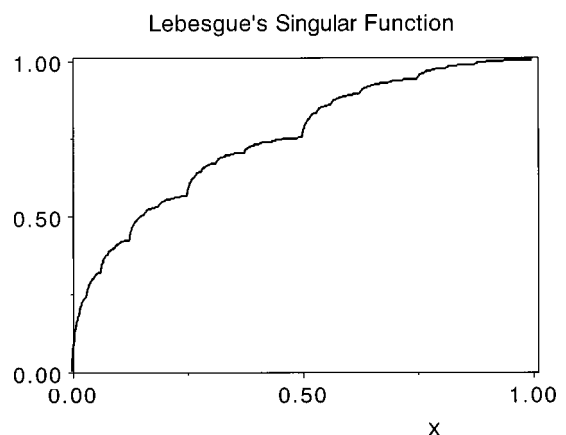


FIG. 4. Lebesgue's singular function for $\alpha=0.75$.

a singular function of x , and there exists an interesting relation between the parameter derivative of $f_\alpha(x)$ and the Takagi function

$$\frac{d}{d\alpha} f_\alpha(x)|_{\alpha=1/2} = 2T(x). \tag{9}$$

In 1957, de Rham found that the Weierstrass function, the Takagi function and Lebesgue's singular function as well as other singular functions are fully characterized as a unique solution f of a functional equation

$$f(x) = \mathcal{F}f(x) + g(x), \tag{10}$$

where $g(x)$ is a given function and \mathcal{F} is a contraction mapping with $\mathcal{F}0=0$ defined in the space of bounded functions on the unit interval $[0,1]$.³⁹ Then, he generalized the functional equation (10) to describe fractal continuous curves such as the ones by Koch or Lévy. The contraction mapping means that the inequality

$$\|\mathcal{F}f_1 - \mathcal{F}f_2\| \leq \lambda \|f_1 - f_2\|$$

holds for some constant $0 < \lambda < 1$ and for any functions f_1 and f_2 , where the function norm $\|\cdot\|$ is defined as the supremum: $\|f\| \equiv \sup_{[0,1]} |f(x)|$. The existence of a unique solution of de Rham's functional equation (10) immediately follows from Banach's contraction mapping theorem⁴⁹ and the fact that the mapping $f \rightarrow \mathcal{F}f + g$ is contraction. Let the mapping $\mathcal{F}_{\alpha,\beta}$ be

$$\mathcal{F}_{\alpha,\beta} f(x) \equiv \begin{cases} \alpha f(2x), & x \in [0, 1/2] \\ \beta f(2x-1), & x \in (1/2, 1] \end{cases} \tag{11}$$

then the Weierstrass function $W_{a,2}$, the Takagi function $T(x)$ and Lebesgue's singular function $f_\alpha(x)$ are the unique solution of (10), respectively, for $\alpha = \beta = a$, $g(x) = \cos \pi x$; for $\alpha = \beta = 1/2$, $g(x) = |x - [x + 1/2]|$; and for $\beta = 1 - \alpha$, $g(x) = \alpha \theta(x - 1/2)$ with θ the step function.³⁹ For more information on singular functions, see Refs. 47 and 51.

Before closing this section, we remark on Riemann-Stieltjes integrals with respect to singular functions, which will appear in the next section. Suppose $f(x)$ is of bounded variation and $\varphi(x)$ is continuous. Then, the Riemann-Stieltjes integral of φ with respect to f is defined by the limit

$$\int_a^b \varphi(x) df(x) \equiv \lim_{\max(x_k - x_{k-1}) \rightarrow 0} \sum_{k=1}^n \varphi(\xi_k) \{f(x_k) - f(x_{k-1})\}, \tag{12}$$

where $\{x_k\}$ is a partition of $[a,b]$: $a = x_0 < x_1 < \dots < x_n = b$, and $x_{k-1} \leq \xi_k \leq x_k$.^{48,49} Since the formula for the integration by parts

$$\int_a^b \varphi(x) df(x) + \int_a^b f(x) d\varphi(x) = \varphi(b)f(b) - \varphi(a)f(a),$$

holds in general, the Riemann-Stieltjes integral of a function of bounded variation with respect to a continuous function can be defined as above. At first sight, the evaluation of the Riemann-Stieltjes integral (12) seems to be difficult. But, for

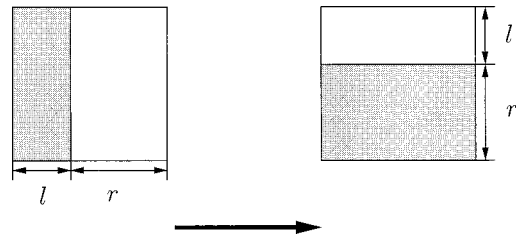


FIG. 5. Schematic representation of the nonconservative reversible Baker map. The shaded rectangle is expanded and the rest is contracted.

a class of functions obeying de Rham's equation, it is not the case. As an example, consider the Fourier-Laplace transformation of $f_\alpha(x)$

$$I(\eta) \equiv \int_0^1 e^{i\eta x} df_\alpha(x).$$

By dividing the integral into the ones over $[0, 1/2]$ and $[1/2, 1]$ and changing the variable x to $x/2$ and $(x+1)/2$, respectively, we have the recursion relation

$$\begin{aligned} I(\eta) &= \int_0^1 e^{i\eta x/2} df_\alpha(x/2) + \int_0^1 e^{i\eta(x+1)/2} df_\alpha((x+1)/2) \\ &= \alpha \int_0^1 e^{i\eta x/2} df_\alpha(x) + (1-\alpha)e^{i\eta/2} \int_0^1 e^{i\eta x/2} df_\alpha(x) \\ &= \{\alpha + (1-\alpha)e^{i\eta/2}\} I(\eta/2), \end{aligned}$$

where the de Rham equation (10) is used in the second equality. Because $I(0) = \int_0^1 df_\alpha(x) = f_\alpha(1) = 1$, the above recursion relation gives^{38,47,51,52}

$$\int_0^1 e^{i\eta x} df_\alpha(x) = I(\eta) = \prod_{n=1}^\infty \{\alpha + (1-\alpha)e^{i\eta/2^n}\}. \tag{13}$$

Note that, for $\alpha \neq 1/2$, $I(2^m \pi) = I(2\pi)(\neq 0)$ ($m=0,1,\dots$) and, hence, the Fourier-Laplace transformation of f_α does not satisfy the Riemann-Lebesgue lemma: $\lim_{\eta \rightarrow \infty} I(\eta) \neq 0$, that again implies the singular nature of f_α .

Further we notice that the formula (13) and the Hata-Yamaguti relation (9) relate the Lebesgue's singular function and the Takagi function to the Weierstrass functions

$$\begin{aligned} f_\alpha(x) &= \alpha - \sum_{s>0:\text{odd}} \frac{\text{Im } I(2\pi s)}{\pi s} W_{1/2,2}(2sx) \\ &\quad + \sum_{s>0:\text{odd}} \frac{\text{Re } I(2\pi s) - 1}{\pi s} \tilde{W}_{1/2,2}(2sx), \\ T(x) &= \frac{1}{2} - \sum_{s>0:\text{odd}} \frac{2}{\pi^2 s^2} W_{1/2,2}(2sx), \end{aligned}$$

where the sums run over positive odd integers, $W_{1/2,2}(x)$ is the Weierstrass function (6) and $\tilde{W}_{1/2,2}(x)$ is a singular function obtained from (6) by replacing \cos to \sin .

III. SRB MEASURE FOR A NONCONSERVATIVE REVERSIBLE BAKER MAP

In thermostated systems, dynamics is nonconservative due to the damping force mimicking a heat reservoir and

thus, the forward time evolution is different from the backward time evolution. However, it is time reversible.⁴⁻¹² It is therefore interesting to see how these two features are compatible, that we shall study with a simple map. One of the simplest nonconservative systems which have time reversal symmetry is given by (Fig. 5)

$$\Phi(x,y) = \begin{cases} (x/l, r y), & x \in [0, l] \\ ((x-l)/r, ly+r), & x \in (l, 1], \end{cases} \quad (14)$$

where $l+r=1$ and $0 < l < 1$. The map is nonconservative since its Jacobian takes r/l for $x \in [0, l]$ and l/r for $x \in (l, 1]$, both of which are different from 1. But the map has a time reversal symmetry represented by an involution $I(x,y) \equiv (1-y, 1-x)$: $I\Phi I = \Phi^{-1}$. The Frobenius-Perron equation governing the time evolution of distribution functions (with respect to the Lebesgue measure) is given by

$$\begin{aligned} \rho_{t+1}(x,y) &= U\rho_t(x,y) \\ &\equiv \int dx' dy' \delta[(x,y) - \Phi(x',y')] \rho_t(x',y') \\ &= \begin{cases} \frac{l}{r} \rho_t(lx, y/r), & y \in [0, r] \\ \frac{r}{l} \rho_t(rx+l, (y-r)/l), & y \in (r, 1], \end{cases} \end{aligned} \quad (15)$$

where U stands for the Frobenius-Perron operator defined by the second equality and $\delta[\cdot]$ is the two-dimensional delta function. Since the map Φ is not conservative, the numerical factors r/l and l/r different from 1 appear in the last expression.

A. An SRB measure for the forward time evolution

First we explain our algorithm to construct SRB measures and apply it to the forward time evolution. Our goal is to show that an expectation value of the dynamical function $\varphi(x,y)$ with respect to ρ_t converges for $t \rightarrow +\infty$ to an expectation with respect to a singular measure given below, when the initial distribution function ρ_0 is continuously differentiable in x , and a dynamical function φ is continuously differentiable in y and continuous in x . We remark that the convergence rate is controlled by the smoothness of the initial distribution function and the dynamical function, and the condition given above is sufficient for the exponential convergence.

The first step in our explicit construction of the singular measure is to express the expectation value by the partially integrated distribution function, i.e.,

$$\begin{aligned} &\int_{[0,1]^2} \varphi(x,y) \rho_t(x,y) dx dy \\ &= \int_{[0,1]^2} \varphi(x,y) d_y P_t(x,y) dx \\ &= \int_0^1 \varphi(x,1) P_t(x,1) dx - \int_{[0,1]^2} \partial_y \varphi(x,y) P_t(x,y) dx dy, \end{aligned} \quad (16)$$

where $P_t(x,y) \equiv \int_0^y dy' \rho_t(x,y')$ is the partially integrated

distribution function, the symbol d_y stands for the Riemann-Stieltjes integral of P_t only with respect to the variable y ⁵³ and the integration by parts is used in the second equality. The evolution equation for P_t can be obtained easily from Eq. (15)

$$P_{t+1}(x,y) = \begin{cases} lP_t(lx, y/r), & y \in [0, r] \\ rP_t(rx+l, (y-r)/l) + lP_t(lx, 1), & y \in (r, 1]. \end{cases} \quad (17)$$

Partial integration of the distribution function changes the prefactors from r/l and l/r , respectively, to r and l , which are strictly less than unity. Because of this, the evolution equation (17) is similar to de Rham's functional equation (10).

The next step in our construction of the singular measure is to calculate the long time limit of P_t . Putting $y=1$ in Eq. (17), we obtain

$$P_{t+1}(x,1) = rP_t(rx+l, 1) + lP_t(lx, 1). \quad (18)$$

Note that this is nothing but the Frobenius-Perron equation for a one-dimensional chaotic map (strictly speaking, an exact map cf. Ref. 54) $Sx = x/l$ (for $x \in [0, l]$) and $Sx = (x-l)/r$ (for $x \in (l, 1]$), which admits the Lebesgue measure as the invariant measure. Therefore, the normalization integral $\int_0^1 dx P_t(x,1)$ is invariant

$$\begin{aligned} \int_0^1 dx P_t(x,1) &= \int_0^1 dx P_{t-1}(x,1) \\ &= \dots = \int_0^1 dx P_0(x,1), \end{aligned}$$

and is equal to the long time limit of the partially integrated distribution function $P_t(x,1)$.⁵⁴ As will be shown in Appendix A, the convergence rate is $\lambda \equiv \max(l,r)$

$$\begin{aligned} P_t(x,1) &= \int_0^1 dx' P_0(x',1) + O(\lambda^t) \\ &= \int_{[0,1]^2} \rho_0(x',y') dx' dy' + O(\lambda^t). \end{aligned} \quad (19)$$

In order to proceed with the calculation, one needs the following lemma. (For its proof, see Appendix A.)

Lemma: Let \mathcal{F} be a linear contraction mapping with the contraction constant $0 < \lambda < 1$. And let $g_t \equiv g^{(0)} + \nu^t g^{(1)} + g_t^{(2)}$ be a given function where ν is a constant satisfying $\lambda < |\nu| \leq 1$, and $g_t^{(2)} = O(\lambda^t)$. Then, the solution of the functional equation

$$f_{t+1} = \mathcal{F}f_t + g_t, \quad (20)$$

is given by

$$f_t = f_\infty^{(0)} + \nu^t f_\infty^{(1)} + O(t\lambda^t), \quad (21)$$

where $f_\infty^{(0)}$ and $f_\infty^{(1)}$ are the unique solutions of the following fixed point equations:

$$f_\infty^{(0)} = \mathcal{F}f_\infty^{(0)} + g^{(0)}, \quad (22)$$

$$f_\infty^{(1)} = \frac{1}{\nu} \mathcal{F}f_\infty^{(1)} + \frac{g^{(1)}}{\nu}. \quad (23)$$

Now we go back to the Eq. (17) for $P_t(x,y)$, which can be rewritten as

$$P_{t+1} = \mathcal{F}P_t + g_t, \tag{24}$$

where the contraction mapping \mathcal{F} and a function g_t are, respectively, given by

$$\mathcal{F}P(x,y) = \begin{cases} lP(lx,y/r), & y \in [0,r] \\ rP(rx+l,(y-r)/l), & y \in (r,1] \end{cases} \tag{25}$$

and

$$g_t(x,y) \equiv \begin{cases} 0, & y \in [0,r] \\ lP_t(lx,1), & y \in (r,1] \end{cases} \\ = \bar{g}^{(0)}(y) \int_{[0,1]^2} \rho_0(x',y') dx' dy' + O(\lambda^t), \tag{26}$$

with

$$\bar{g}^{(0)}(y) = \begin{cases} 0, & y \in [0,r] \\ l, & y \in (r,1]. \end{cases} \tag{27}$$

The contraction constant of the mapping \mathcal{F} is $\lambda = \max(l,r)$ and Eq. (19) is used to derive the left-hand side of Eq. (26). As the Eqs. (24), (25), (26) and (27) satisfy the condition of the lemma and the contraction mapping \mathcal{F} given by (25) is linear, the lemma implies

$$P_t(x,y) = F_t(y) \int_{[0,1]^2} \rho_0(x',y') dx' dy' + O(t\lambda^t), \tag{28}$$

where $F_t(y)$ is the unique solution of de Rham's functional equation

$$F_t(y) = \mathcal{F}F_t(y) + \bar{g}^{(0)}(y) \\ = \begin{cases} lF_t(y/r), & y \in [0,r] \\ rF_t((y-r)/l) + l, & y \in (r,1]. \end{cases} \tag{29}$$

By substituting Eq. (28) into Eq. (16) and employing the integration by parts, we have

$$\int_{[0,1]^2} \varphi(x,y) \rho_t(x,y) dx dy \\ = \int_{[0,1]^2} \varphi(x,y) dx dF_t(y) \\ \times \int_{[0,1]^2} \rho_0(x',y') dx' dy' + O(t\lambda^t). \tag{30}$$

We remind the reader that Eq. (30) holds for any continuous function $\varphi(x,y)$ and any integrable function $\rho_0(x,y)$ which are continuously differentiable, respectively, in y and x . If ρ_0 is normalized with respect to the Lebesgue measure, Eq. (30) gives

$$\lim_{t \rightarrow \infty} \int_{[0,1]^2} \varphi(x,y) \rho_t(x,y) dx dy = \int_{[0,1]^2} \varphi(x,y) dx dF_t(y). \tag{31}$$

This shows that the physical measure μ_{ph} of the system is given by

$$\mu_{ph}([0,x] \times [0,y]) = xF_t(y). \tag{32}$$

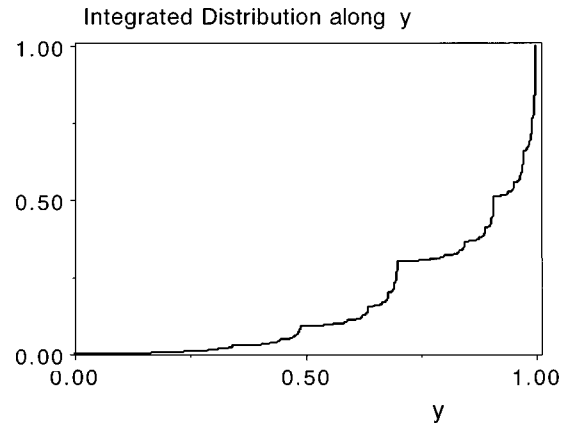


FIG. 6. Partially integrated distribution of the physical measure μ_{ph} along y for the nonconservative reversible map with $l=0.3$.

Clearly it is absolutely continuous with respect to the Lebesgue measure along the expanding x -direction and, thus is an SRB measure. As studied in Ref. 38, the function F_l is nondecreasing, has zero derivatives almost everywhere except for $r=l=1/2$ with respect to the Lebesgue measure and is Hölder continuous with exponent $\delta = -\ln \max(l,r) / \ln \min(l^{-1},r^{-1}) = 1$ (i.e., $|F_l(x) - F_l(y)| \leq A|x - y|$). The graph of F_l is a fractal (Fig. 6), but its fractal dimension is $D=1$ as a result of the Besicovich-Ursell inequality:⁵⁶ $1 \leq D \leq 2 - \delta = 1$. Moreover, the one-dimensional measure defined by F_l is a multifractal two-scale Cantor measure, the dimension spectrum D_q ($-\infty < q < +\infty$) of which is given as the solution of^{38,55}

$$l^q r^{(1-q)D_q} + r^q l^{(1-q)D_q} = 1.$$

There exists an interesting relation between F_l and Lebesgue's singular function f_α

$$F_l(y) = f_l \circ f_r^{-1}(y),$$

which immediately follows from the fact that the right hand side obeys the same functional equation as F_l . Note that, since f_r is continuous and strictly increasing, the inverse f_r^{-1} exists and is also strictly increasing. As a composite function of two strictly increasing functions f_l and f_r^{-1} , F_l is also strictly increasing. Because of these singular properties of F_l , the physical measure μ_{ph} is singular along the contracting y -direction.

The physical measure μ_{ph} is mixing with respect to the map Φ . Indeed, by considering $P_t(x,y) = \int_0^y \rho_t(x,y') dF_l(y')$ instead of $P_t(x,y) = \int_0^y \rho_t(x,y') dy'$ and following exactly the same arguments as above, one obtains

$$\lim_{t \rightarrow \infty} \int_{[0,1]^2} \varphi(x,y) \rho_t(x,y) dx dF_l(y) \\ = \int_{[0,1]^2} \varphi(x,y) dx dF_l(y), \tag{33}$$

provided that $\varphi(x,y)$ and $\rho_0(x,y)$ are continuously differentiable, respectively, in y and x and that ρ_0 is normalized: $\int \rho_0(x,y) dx dF_l(y) = 1$. By using this fact, the Lyapunov exponents can be analytically calculated as follows: The

Jacobian matrix for Φ is diagonal, and the logarithm of the component along the expanding x direction, the local expanding rate $\Lambda_x(x,y)$, is

$$\Lambda_x(x,y) = \begin{cases} -\ln l, & x \in [0,l] \\ -\ln r, & x \in (l,1]. \end{cases}$$

The Lyapunov exponent along the x direction (the positive Lyapunov exponent) is defined as the average of $\Lambda_x(x,y)$ over an orbit starting from some initial point (x,y) . Since the system is ergodic, the Lyapunov exponent can be obtained from Eq. (2) using the measure μ_{ph}

$$\begin{aligned} \lambda_x(\Phi, \mu_{\text{ph}}) &\equiv \lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=0}^T \Lambda_x(\Phi^t(x,y)) \\ &= \int_{[0,1]^2} \Lambda_x(x,y) dx dF_l(y) \\ &= -l \ln l - r \ln r. \end{aligned} \tag{34}$$

Similarly, the Lyapunov exponent along the y -direction (the negative Lyapunov exponent) is

$$\lambda_y(\Phi, \mu_{\text{ph}}) = l \ln r + r \ln l. \tag{35}$$

The sum of the two Lyapunov exponents is negative

$$\lambda_x(\Phi, \mu_{\text{ph}}) + \lambda_y(\Phi, \mu_{\text{ph}}) = (r-l) \ln\left(\frac{l}{r}\right) < 0.$$

Hence, areas are contracted on average by the map Φ and the map possesses an attractor A . From Eq. (31), one finds that the attractor A is the support of the SRB measure μ_{ph} . When $l \neq 1/2$, the two-dimensional Lebesgue measure of the attractor A is zero since the function F_l has zero derivatives almost everywhere with respect to the Lebesgue measure. Moreover, according to Young's formula⁵⁷ (which is the Kaplan-Yorke formula^{16,17,30,58} for two-dimensional ergodic systems), the information dimension of A is given by

$$D_1 = 1 + \frac{\lambda_x(\Phi, \mu_{\text{ph}})}{|\lambda_y(\Phi, \mu_{\text{ph}})|} = 1 + \left| \frac{l \ln l + r \ln r}{l \ln r + r \ln l} \right| < 2,$$

which is less than two. Therefore, A is a fractal set. On the other hand, since the function F_l is strictly increasing, for any nonempty rectangle $[x_0, x_0 + \epsilon] \times [y_0, y_0 + \epsilon']$ ($\epsilon > 0$ and $\epsilon' > 0$), we have

$$\begin{aligned} \mu_{\text{ph}}([x_0, x_0 + \epsilon] \times [y_0, y_0 + \epsilon']) \\ = \epsilon \{F_l(y_0 + \epsilon') - F_l(y_0)\} > 0, \end{aligned}$$

which implies that $A \cap [x_0, x_0 + \epsilon] \times [y_0, y_0 + \epsilon'] \neq \emptyset$ and, hence, the attractor A is dense in the whole phase space $[0,1]^2$. This phase-space structure is in contrast to the one of a dissipative system usually studied³⁰ (see also the next section), where an attractor is a Cantor-like set.

B. An SRB measure for the backward time evolution

Now we consider the backward time evolution. In the same way as the forward evolution, we find that another partially integrated distribution function $\bar{P}_t(x,y) = \int_0^x dx' \rho_t(x',y)$ converges for $t \rightarrow -\infty$ and we have

$$\begin{aligned} &\int_{[0,1]^2} \varphi(x,y) \rho_t(x,y) dx dy \\ &= \int_{[0,1]^2} \varphi(x,y) d\bar{F}_r(x) dy \int_{[0,1]^2} \rho_0(x',y') dx' dy' \\ &\quad + O(|t| |\lambda|^{|t|}), \end{aligned} \tag{36}$$

where $t=0, -1, -2, \dots$ and a singular function \bar{F}_r is given by $\bar{F}_r(x) = 1 - F_l(1-x)$. As before, Eq. (36) implies that the asymptotic physical measure $\bar{\mu}_{\text{ph}}$ is given by $\bar{\mu}_{\text{ph}}([0,x] \times [0,y]) \equiv \bar{F}_r(x)y$. The measure $\bar{\mu}_{\text{ph}}$ is then absolutely continuous with respect to the Lebesgue measure along y direction and singular along x -direction. This corresponds to the fact that the expanding and contracting directions are interchanged for the backward motion. Actually, the measure $\bar{\mu}_{\text{ph}}$ is precisely the one obtained from μ_{ph} via the time reversal operation $I: \bar{\mu}_{\text{ph}} = I\mu_{\text{ph}}$. The measure $\bar{\mu}_{\text{ph}}$ is again mixing with respect to the backward time evolution Φ^{-t} ($t=0, 1, \dots$). And the Lyapunov exponents are calculated as the phase space averages of the local scaling rates for the inverse map Φ^{-1} with respect to $\bar{\mu}_{\text{ph}}$. For example, the positive Lyapunov exponent is the $\bar{\mu}_{\text{ph}}$ -average of the local expanding rate: $\bar{\Lambda}_y(x,y) = -\theta(r-y) \ln r - \theta(y-r) \ln l$ and is equal to that for the original map Φ

$$\begin{aligned} \lambda_y(\Phi^{-1}, \bar{\mu}_{\text{ph}}) &= \int_{[0,1]^2} \bar{\Lambda}_y(x,y) d\bar{F}_r(x) dy \\ &= -r \ln r - l \ln l = \lambda_x(\Phi, \mu_{\text{ph}}). \end{aligned} \tag{37}$$

The negative Lyapunov exponents of the two maps are also the same

$$\lambda_x(\Phi^{-1}, \bar{\mu}_{\text{ph}}) = \lambda_y(\Phi, \mu_{\text{ph}}). \tag{38}$$

The equality of Lyapunov exponents for (Φ, μ_{ph}) and $(\Phi^{-1}, \bar{\mu}_{\text{ph}})$ is a general consequence of the time reversal symmetry of the system.

We notice that the natural invariant measure $\bar{\mu}_{\text{ph}}$ of Φ^{-1} is also invariant under Φ as follows straightforwardly from the reversibility of Φ

$$\bar{\mu}_{\text{ph}}(\Phi^{-1}\{[0,x] \times [0,y]\}) = \bar{\mu}_{\text{ph}}([0,x] \times [0,y]),$$

is equivalent to

$$\bar{\mu}_{\text{ph}}([0,x] \times [0,y]) = \bar{\mu}_{\text{ph}}(\Phi\{[0,x] \times [0,y]\}). \tag{39}$$

That is, we may think of $\bar{\mu}_{\text{ph}}$ as a repelling measure for Φ , in the sense that, while it is indeed invariant, any slight deviations from this measure, if they are absolutely continuous with respect to the Lebesgue measure, will evolve toward the measure μ_{ph} for the attractor for Φ [cf. Eq. (31)]. In short, we find, for a nonconservative reversible map Φ , different SRB measures μ_{ph} and $\bar{\mu}_{\text{ph}}$ for the forward and backward time evolutions, respectively. And for each time evolution, one plays a role of an attracting measure and the other a role of a repelling measure in the sense just explained. This observation is a key element of the compatibility between dynamical reversibility and irreversible behavior of statistical ensembles. Indeed, when the dynamics is reversible and sta-

tistical ensembles irreversibly approach a stationary ensemble μ_{ph} for $t \rightarrow +\infty$, there should exist another stationary ensemble $\bar{\mu}_{\text{ph}}$ which is obtained from μ_{ph} by the time reversal operation. However, the new stationary ensemble $\bar{\mu}_{\text{ph}}$ is repelling and, thus its existence is compatible with the irreversible behavior of statistical ensembles.

Further, the measure $\bar{\mu}_{\text{ph}}$ is mixing with respect to the forward time evolution Φ^t ($t=0,1,\dots$). Then it is interesting to investigate the relation between Lyapunov exponents and the Kolmogorov-Sinai (KS) entropy for $(\Phi, \bar{\mu}_{\text{ph}})$. The Lyapunov exponents for $(\Phi, \bar{\mu}_{\text{ph}})$ are easily found to be

$$\lambda_x(\Phi, \bar{\mu}_{\text{ph}}) = -r \ln l - l \ln r, \tag{40}$$

and

$$\lambda_y(\Phi, \bar{\mu}_{\text{ph}}) = r \ln r + l \ln l. \tag{41}$$

It is useful to note that the positive (negative) Lyapunov exponent of $\bar{\mu}_{\text{ph}}$ can be found by changing the sign of the negative (positive) Lyapunov exponent of μ_{ph} . This fact is widely used in the analysis of the Lyapunov spectrum of thermostated many particle systems.^{4,7,8} In our case this result follows from the observations that

$$\lambda_x(\Phi, \bar{\mu}_{\text{ph}}) = -\lambda_y(\Phi, \mu_{\text{ph}}) \tag{42}$$

and

$$\lambda_y(\Phi, \bar{\mu}_{\text{ph}}) = -\lambda_x(\Phi, \mu_{\text{ph}}). \tag{43}$$

Thus the Lyapunov spectrum changes sign under the exchange of μ_{ph} and $\bar{\mu}_{\text{ph}}$. Now we turn to a calculation of the KS-entropy. First we note that the KS-entropy of (Φ, μ_{ph}) is

$$h_{\text{KS}}(\Phi, \mu_{\text{ph}}) = \lambda_x(\Phi, \mu_{\text{ph}}), \tag{44}$$

which follows from the Pesin's identity,¹⁶ since μ_{ph} is the SRB measure for the map Φ , and which can also be computed directly from the entropy of the generating partition formed by the two elements $0 \leq x \leq l$ and $l \leq x \leq 1$. From the same partition, it is readily seen that the entropy of $(\Phi, \bar{\mu}_{\text{ph}})$ is also

$$h_{\text{KS}}(\Phi, \bar{\mu}_{\text{ph}}) = \lambda_x(\Phi, \mu_{\text{ph}}). \tag{45}$$

Then the difference between the positive Lyapunov exponent and the KS-entropy for $(\Phi, \bar{\mu}_{\text{ph}})$ is

$$\lambda_x(\Phi, \bar{\mu}_{\text{ph}}) - h_{\text{KS}}(\Phi, \bar{\mu}_{\text{ph}}) = -\lambda_y(\Phi, \mu_{\text{ph}}) - \lambda_x(\Phi, \mu_{\text{ph}}) \geq 0, \tag{46}$$

which is strictly positive for $l \neq 1/2$ since the right hand side is just the phase space contraction rate. Therefore, the mixing system $(\Phi, \bar{\mu}_{\text{ph}})$ violates Pesin's identity, but satisfies Ruelle's inequality,¹⁶ as it should be since the measure $\bar{\mu}_{\text{ph}}$ is not an SRB measure for Φ .

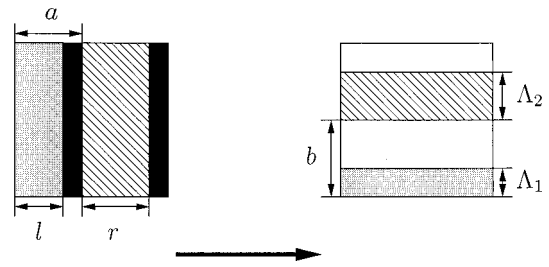


FIG. 7. Schematic representation of the Baker map with a Cantor-like invariant set. The points in the black rectangles are removed. The other shaded rectangles are mapped onto the corresponding ones.

IV. SRB MEASURE FOR A BAKER MAP WITH A CANTOR-LIKE INVARIANT SET

Transport properties are also studied for open Hamiltonian systems with a flux boundary condition¹³⁻¹⁵ or with an absorbing boundary condition.^{12,20-26} Breyman, Tél and Vollmer used an open nonconservative system to study an interrelation between the thermostated systems approach and the open systems approach.²⁹ Further, a model used by Kaufmann, Lustfeld, Németh and Szépfalusky⁵⁹ to investigate deterministic transient diffusion is a nonconservative open system. One of the important features of those open systems is the existence of escape. So we investigate how escape affects the physical measure, by using a Baker map with a Cantor-like invariant set.^{10,30,58} The map is defined on the unit square $[0,1]^2$ (Fig. 7):

$$\Psi(x,y) = \begin{cases} (x/l, \Lambda_1 y), & x \in [0,l] \\ ((x-a)/r, \Lambda_2 y + b), & x \in [a, a+r], \end{cases} \tag{47}$$

where $0 < l \leq a$, $0 < r \leq 1-a$, $0 < \Lambda_1 \leq b$ and $0 < \Lambda_2 \leq 1-b$. Here we introduce an escape for points $x \in (l,a) \cup (a+r,1]$ and an inhomogeneity. For $\Lambda_1 < l$ and $\Lambda_2 > r$, or for $\Lambda_1 > l$ and $\Lambda_2 < r$, the map Ψ is partially attractive and partially repelling as discussed in the previous section and, for $\Lambda_1 = r$, $\Lambda_2 = l$, and $l+r=1$, it is reduced to the previous model. It is everywhere attractive for $\Lambda_1 < l$ and $\Lambda_2 < r$, is conservative for $\Lambda_1 = l$ and $\Lambda_2 = r$ and is everywhere repelling for $\Lambda_1 > l$ and $\Lambda_2 > r$. Note that the last case is possible only when there exists an escape: $l+r < 1$.

We show that when the initial distribution function ρ_0 is continuously differentiable in x , and a dynamical function φ is continuously differentiable in y and continuous in x , an expectation value of the dynamical function $\varphi(x,y)$ with respect to ρ_t decays exponentially

$$\begin{aligned} & \int_{[0,1]^2} \varphi(x,y) \rho_t(x,y) dx dy \\ &= \nu^t \int_{[0,1]^2} \varphi(x,y) dx dG(y) \\ & \quad \times \int_{[0,1]^2} \rho_0(x,y) dH(x) dy + O(t\lambda'^t), \end{aligned} \tag{48}$$

where the decay rate is equal to the remainder volume per iteration: $\nu = l+r$, $\lambda' = \max(l,r) (\leq \nu)$. The functions G and H are (possibly) singular functions defined as the unique solutions of de Rham equations

$$G(y) = \begin{cases} \frac{l}{l+r} G\left(\frac{y}{\Lambda_1}\right), & y \in [0, \Lambda_1] \\ \frac{l}{l+r}, & y \in (\Lambda_1, b) \\ \frac{r}{l+r} G\left(\frac{y-b}{\Lambda_2}\right) + \frac{l}{l+r}, & y \in [b, \Lambda_2 + b] \\ 1, & y \in (\Lambda_2 + b, 1] \end{cases} \quad (49)$$

and

$$H(x) = \begin{cases} \frac{l}{l+r} H\left(\frac{x}{l}\right), & x \in [0, l] \\ \frac{l}{l+r}, & x \in (l, a) \\ \frac{r}{l+r} H\left(\frac{x-a}{r}\right) + \frac{l}{l+r}, & x \in [a, a+r] \\ 1, & x \in (a+r, 1]. \end{cases} \quad (50)$$

The derivation of Eq. (48) is outlined in Appendix B. Now we investigate the implications of Eq. (48) in cases without and with escape separately.

A. Baker map without escape

Consider first the case where there is no escape ($l+r=1$). The unit square is then contracted onto a set which is a direct product of a Cantor set in the y -direction and the unit interval, $[0,1]$, in the x -direction. This direct product set is nothing but the strange attractor of Ψ , whose Hausdorff dimension $1 < H_D < 2$. In this case, $H(x)=x$ and (48) reduces to an expression similar to (36) for the previous example. And the function G defines an invariant mixing measure μ_{ph}

$$\mu_{ph}([0,x) \times [0,y)) = xG(y). \quad (51)$$

The measure μ_{ph} is the one which provides long-term averages of dynamical functions and thus, is a physical measure. Indeed, when $\int \rho_0(x,y) dx dy = 1$, (48) gives

$$\begin{aligned} \langle \varphi \rangle_t &\equiv \int_{[0,1]^2} \varphi(x,y) \rho_t(x,y) dx dy \\ &= \int_{[0,1]^2} \varphi(x,y) dx dG(y) + O(t\{\lambda'\}^t). \end{aligned} \quad (52)$$

Since μ_{ph} is smooth (more precisely absolutely continuous with respect to the Lebesgue measure) along the expanding coordinate x , it is an SRB measure. As shown in Fig. 8, the graph of $G(y)$ is a typical devil's staircase and the measure μ_{ph} is singular. Note that the support of μ_{ph} is the strange attractor of Ψ .

B. Baker map with escape

When $l+r < 1$, almost all the points escape the unit square and there appears a fractal repeller, which is singular both in expanding and contracting directions. Then, the invariant measure supported by the fractal repeller is different from the physical measure defined by Eq. (5), which is not invariant but conditionally invariant under Ψ .^{23,24}

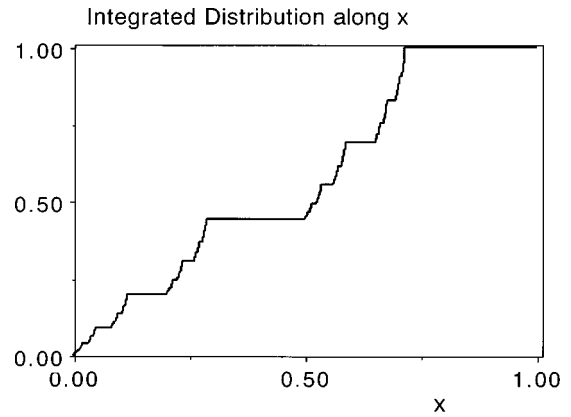


FIG. 8. Partially integrated distribution of the physical measure μ_{ph} along x for the Baker map with a Cantor-like invariant set. The parameters are $\Lambda_1 = 0.4$, $\Lambda_2 = 0.3$, $b = 0.5$ and $l/r = 0.8$.

First we consider the physical measure. The formula (48) holds even when $l+r < 1$. By setting $\varphi \equiv 1$ in (48), the renormalization factor $\int \rho_t(x,y) dx dy$ is found to be

$$\begin{aligned} \int_{[0,1]^2} \rho_t(x,y) dx dy &= v^t \int_{[0,1]^2} \rho_0(x,y) dH(x) dy \\ &\quad + O(t\lambda'^t). \end{aligned}$$

Hence, the expectation value of φ at time t is

$$\begin{aligned} \langle \varphi \rangle_t &\equiv \frac{\int_{[0,1]^2} \varphi(x,y) \rho_t(x,y) dx dy}{\int_{[0,1]^2} \rho_t(x,y) dx dy} \\ &= \int_{[0,1]^2} \varphi(x,y) dx dG(y) + O\left(t\left\{\frac{\lambda'}{v}\right\}^t\right). \end{aligned} \quad (53)$$

This implies that the physical measure defined by Eq. (5) is identical to that for the Baker map without escape

$$\mu_{ph}([0,x) \times [0,y)) = xG(y), \quad (54)$$

provided that the ratio l/r is common in two cases. As in the previous case, the physical measure μ_{ph} is singular and is supported by a direct product of a Cantor set C_1 along y and the unit interval along x : $[0,1] \times C_1$, which is the unstable manifold of the fractal repeller. Since it is smooth along the expanding x -direction, it is an SRB-like measure (in the sense that, although not invariant, it is smooth along the expanding direction).

We observe that the support of μ_{ph} is the unstable manifold of the repeller and is not an invariant set with respect to Ψ . Accordingly, the physical measure μ_{ph} is not an invariant measure. Indeed, the measure μ_{ph} satisfies

$$\mu_{ph}(\Psi^{-1}\{[0,x) \times [0,y)\}) = (l+r)\mu_{ph}([0,x) \times [0,y)), \quad (55)$$

which implies that μ_{ph} shrinks as time goes on. This can be seen immediately from the functional equation for G . For example, when $y \in [0, \Lambda_1]$,

$$\begin{aligned} \mu_{ph}(\Psi^{-1}\{[0,x) \times [0,y)\}) &= \mu_{ph}([0, lx) \times [0, y/\Lambda_1)) = lxG(y/\Lambda_1) \\ &= (l+r)xG(y) = (l+r)\mu_{ph}([0,x) \times [0,y)). \end{aligned}$$

Then, since the measure μ_{ph} satisfies

$$\mu_{\text{ph}}(E) = \frac{\mu_{\text{ph}}(\Psi^{-1}E)}{\mu_{\text{ph}}(\Psi^{-1}[0,1]^2)},$$

for any Borel set $E \subset [0,1]^2$, it is conditionally invariant.^{23,24}

Now we turn to an invariant measure μ_{in} on the repeller, which is defined by

$$\mu_{\text{in}}([0,x] \times [0,y]) = H(x)G(y). \quad (56)$$

The invariance can be seen straightforwardly from the functional equations for G and H . For example, when $y \in [b, \Lambda_2 + b]$, one has

$$\begin{aligned} \mu_{\text{in}}(\Psi^{-1}\{[0,x] \times [0,y]\}) &= \mu_{\text{in}}([0,lx] \times [0,1]) \cup [a, rx+a] \times [0, (y-b)/\Lambda_2]) \\ &= \mu_{\text{in}}([0,lx] \times [0,1]) + \mu_{\text{in}}([a, rx+a] \times [0, (y-b)/\Lambda_2]) \\ &= H(lx)G(1) + \{H(rx+a) - H(a)\}G((y-b)/\Lambda_2) \\ &= H(x) \frac{l}{l+r} + \frac{r}{l+r} H(x)G((y-b)/\Lambda_2) \\ &= H(x)G(y) = \mu_{\text{in}}([0,x] \times [0,y]). \end{aligned}$$

Since $H(x)$ is a fractal function similar to those discussed in Sec. II, the invariant measure is singular both along the contracting and expanding directions. This can be easily understood as follows: Since the map Ψ eventually transforms the unit square into the unstable manifold of the repeller, which is a direct product of a Cantor set C_1 along y and the unit interval along x : $[0,1] \times C_1$, the measure becomes singular along y . On the other hand, only the orbits starting from the stable manifold of the repeller remain in the unit square and the stable manifold is a direct product set of the unit interval along y and a Cantor set C_2 along x : $C_2 \times [0,1]$. Thus the invariant set is a subset of $C_2 \times [0,1]$. As a result, the invariant measure becomes singular also along x . Actually, the direct product of the two Cantor sets $C_2 \times C_1$ is the fractal repeller of Ψ .

In a similar argument to the derivation of (48), one can show that the invariant measure μ_{in} is mixing with respect to Ψ . As shown in Appendix C, the invariant measure μ_{in} is a Gibbs measure.^{1,12,22-28}

V. A BAKER-TYPE MAP UNDER A FLUX BOUNDARY CONDITION

To illustrate a stationary state for an open system with a flux boundary condition, we study a simple model of a chemical reaction. In simple reactions such as $R \leftrightarrow I \leftrightarrow P$, the reactant R , the intermediate I and the product P consist of the same atoms, and they can be specified by configurations of atoms, or points in the atomic-configuration space. An example of the reaction $R \leftrightarrow I \leftrightarrow P$ is an isomerization, or a change in the conformation of a molecule, such as the ‘‘chair’’ to ‘‘boat’’ transformation of cyclohexane, where R is the chair-shaped isomer, P is the boat-shaped isomer and I is an intermediate unstable isomer, all of which consist of six carbon atoms and twelve hydrogen atoms. Hence, the reaction process can be regarded as a dynamical process where

each trajectory starting from a reactant state to a product state represents an individual reaction $R \rightarrow P$ (cf., Ref. 60, and references therein). A connection between chemical reactions and the escape-rate formalism was investigated by Dorfman and Gaspard²² and a Baker-type model of a chemical reaction was studied by Elskens and Kapral.⁴⁰ In this section, we introduce a Baker-type model of a reaction $R \leftrightarrow I \leftrightarrow P$ and study its statistical properties for two cases: In one case, the system is closed, and in the other case, a flux boundary condition is imposed. Now we begin with the phenomenology.

A. Phenomenology

For a chemical reaction $R \leftrightarrow I \leftrightarrow P$, the discrete-time version of the phenomenological rate equation is

$$\begin{aligned} R_{t+1} &= k_{RR}R_t + k_{RI}I_t, \\ I_{t+1} &= k_{IR}R_t + k_{II}I_t + k_{IP}P_t, \\ P_{t+1} &= k_{PI}I_t + k_{PP}P_t, \end{aligned} \quad (57)$$

where R_t , I_t , and P_t are concentrations of the reactant R , intermediate I , and product P , respectively, and k_{AB} ($A, B = R, I$, or P) are rate coefficients. Since the sum $R_t + I_t + P_t$ is preserved, the rate coefficients satisfy a sum rule

$$\begin{aligned} k_{RR} + k_{IR} &= 1, \\ k_{RI} + k_{II} + k_{PI} &= 1, \\ k_{IP} + k_{PP} &= 1. \end{aligned} \quad (58)$$

The stationary state solution of Eq. (57) is then given by

$$\frac{R_{\text{st}}}{I_{\text{st}}} = \frac{k_{RI}}{k_{IR}}, \quad \frac{P_{\text{st}}}{I_{\text{st}}} = \frac{k_{PI}}{k_{IP}}. \quad (59)$$

Now we consider a stationary solution of Eq. (57) under a flux boundary condition, where the concentrations of the reactant R and the product P are fixed to given values R_{ex} and P_{ex} , respectively. This may be realized, e.g., by introducing source terms to the equations for the reactant and product and by adjusting them so as to keep the values of R_t and P_t constant. Equation (57) is then reduced to

$$I_{t+1} = k_{IR}R_{\text{ex}} + k_{II}I_t + k_{IP}P_{\text{ex}}, \quad (60)$$

which has a stationary solution

$$I_{\text{fl}} = \frac{k_{IR}R_{\text{ex}} + k_{IP}P_{\text{ex}}}{1 - k_{II}}. \quad (61)$$

The deviation $\delta I_t \equiv I_t - I_{\text{fl}}$ of the intermediate concentration from the stationary value decays exponentially

$$\delta I_t = k_{II} \delta I_{t-1} = \dots = k_{II}^t \delta I_0. \quad (62)$$

The stationary state admits a nonvanishing concentration flow from the reactant to the product

$$J_{R \rightarrow P} = k_{PI}I_{\text{fl}} - k_{IP}P_{\text{ex}} = \frac{k_{PI}k_{IR}}{1 - k_{II}}R_{\text{ex}} - \frac{k_{RI}k_{IP}}{1 - k_{II}}P_{\text{ex}}, \quad (63)$$

and, hence, can be regarded as a stationary state under a flux boundary condition.

B. A Baker-type model and a stationary state for a closed system

We introduce a Baker-type model of the reaction process $R \leftrightarrow I \leftrightarrow P$. Microscopic dynamical states of each species R , I , or P are represented by the points in a unit square $(0,1]^2$.

Area preserving asymmetric Baker maps are used to describe the dynamics of the reactant and product states, and a Baker map similar to the one discussed in the previous section is used for the intermediate state dynamics. The model is then defined by (Fig. 9)

$$\Psi'(I:x,y) = \begin{cases} \left(I: \frac{x}{l}, \Lambda_1 y \right), & x \in (0,l] \\ \left(R: \frac{x-l}{a-l}, (b-\Lambda_1)y \right), & x \in (l,a] \\ \left(I: \frac{x-a}{r}, \Lambda_2 y + b \right), & x \in (a,a+r] \\ \left(P: \frac{x-a-r}{1-a-r}, (1-b-\Lambda_2)y \right), & x \in (a+r,1], \end{cases} \tag{64}$$

$$\Psi'(R:x,y) = \begin{cases} \left(I: \frac{x}{b-\Lambda_1}, (b-\Lambda_1)y + \Lambda_1 \right), & x \in (0,b-\Lambda_1] \\ \left(R: \frac{x-b+\Lambda_1}{1-b+\Lambda_1}, (1-b+\Lambda_1)y + b - \Lambda_1 \right), & x \in (b-\Lambda_1,1], \end{cases} \tag{65}$$

$$\Psi'(P:x,y) = \begin{cases} \left(I: \frac{x}{1-b-\Lambda_2}, (1-b-\Lambda_2)y + b + \Lambda_2 \right), & x \in (0,1-b-\Lambda_2] \\ \left(P: \frac{x-1+b+\Lambda_2}{b+\Lambda_2}, (b+\Lambda_2)y + 1 - b - \Lambda_2 \right), & x \in (1-b-\Lambda_2,1], \end{cases} \tag{66}$$

where (x,y) denotes a point in a unit square and α ($\alpha = R, I$, or P) distinguishes different species.

As before, we study the time evolution of a measure starting from an initial measure which is absolutely continuous with respect to the two-dimensional Lebesgue measure. The Frobenius-Perron equation for the density function $\rho_t(\alpha:x,y)$ ($\alpha = R, I$, or P) is obtained from

$$\rho_{t+1}(\alpha:x,y) = \sum_{\beta=R,I,P} \int_{[0,1]^2} dx dy \delta((\alpha:x,y) - \Psi'(\beta:x',y')) \rho_t(\beta:x',y'),$$

where the delta function $\delta((\alpha:x,y) - (\beta:x',y'))$ stands for the product $\delta_{\alpha,\beta} \delta(x-x') \delta(y-y')$. By integrating the Frobenius-Perron equation with respect to y , one obtains the evolution equation for the partially integrated distribution function $Q_t(\alpha:x,y) = \int_0^y dy' \rho_t(\alpha:x,y')$:

$$Q_{t+1}(\alpha:x,y) = \bar{\mathcal{F}} Q_t(\alpha:x,y) + \bar{R}_t(\alpha:x,y), \tag{67}$$

where a contraction mapping $\bar{\mathcal{F}}$ and a functional \bar{R}_t of $Q_t(\alpha:x,1)$ are given in Appendix D [cf., Eqs. (D2), (D3), (D4) and Eqs. (D5), (D6), (D7), respectively].

Now we consider the case where the initial measure is uniform along the expanding x direction and thus,

$Q_0(\alpha:x,y)$ does not depend on x . Then, as seen from the expressions of $\bar{\mathcal{F}}$ and \bar{R}_t , the partially integrated function $Q_t(\alpha:x,y)$ at time t is also independent of x . Particularly, its value $Q_t(\alpha:x,y=1) \equiv Q_t(\alpha)$ at $y=1$ obeys

$$\begin{aligned} Q_{t+1}(R) &= (1-b+\Lambda_1)Q_t(R) + (a-l)Q_t(I), \\ Q_{t+1}(I) &= (b-\Lambda_1)Q_t(R) + (l+r)Q_t(I) \\ &\quad + (1-b-\Lambda_2)Q_t(P), \end{aligned} \tag{68}$$

$$Q_{t+1}(P) = (1-a-r)Q_t(I) + (b+\Lambda_2)Q_t(P).$$

Since the distribution is uniform along the x direction, $Q_t(\alpha)$ is equal to the total probability of finding the system in a species α : $Q_t(\alpha) = \int dx dy \rho_t(\alpha:x,y)$. Therefore, Eq. (68) has exactly the same form as Eq. (57) where the corresponding rate coefficients are given by

$$\begin{aligned} k_{RR} &= 1-b+\Lambda_1, & k_{RI} &= a-l, \\ k_{IR} &= b-\Lambda_1, & k_{II} &= l+r, & k_{IP} &= 1-b-\Lambda_2, \\ k_{PI} &= 1-a-r, & k_{PP} &= b+\Lambda_2. \end{aligned} \tag{69}$$

Note that the rate coefficients given above satisfy the sum rule Eq. (58) and hence, the sum $Q_t(R) + Q_t(I) + Q_t(P)$ is constant in time. We also remark that the rate coefficients

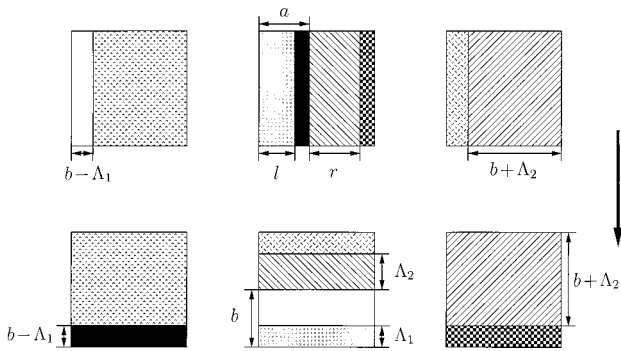


FIG. 9. Schematic representation of the Baker-type map describing a chemical reaction process $R \leftrightarrow I \leftrightarrow P$. Three unit squares describe the dynamical states of the reactant, intermediate and product, respectively. The shaded rectangles are mapped onto the corresponding ones.

(69) admit a simple geometrical interpretation. As an example, we consider k_{RI} , which is the transition probability from the intermediate I to the reactant R . According to the definition of the map Ψ' , a rectangle $(l, a] \times (0, 1]$ moves

from the intermediate states to the reactant states and, thus, its Lebesgue area $(a-l)$ corresponds to the transition probability k_{RI} from I to R , or we have the second equation of (69). The other rate coefficients can be obtained in the same way.

According to the discussion given in the previous subsection, the stationary solution of (68) is

$$Q_{st}(R) = \frac{a-l}{b-\Lambda_1} Q_{st}(I), \quad Q_{st}(P) = \frac{1-a-r}{1-b-\Lambda_2} Q_{st}(I), \tag{70}$$

and the corresponding distribution is given by

$$Q_{st}(R:x, y) = \frac{a-l}{b-\Lambda_1} Q_{st}(I) q_{st}(R:y),$$

$$Q_{st}(I:x, y) = Q_{st}(I) q_{st}(I:y), \tag{71}$$

$$Q_{st}(P:x, y) = \frac{1-a-r}{1-b-\Lambda_2} Q_{st}(I) q_{st}(P:y),$$

where the singular functions $q_{st}(\alpha:y)$ ($\alpha=R, I$, and P) are the unique solutions of de Rham equations

$$q_{st}(I:y) = \begin{cases} l q_{st}\left(I: \frac{y}{\Lambda_1}\right), & y \in (0, \Lambda_1] \\ l + (a-l) q_{st}\left(R: \frac{y-\Lambda_1}{b-\Lambda_1}\right), & y \in (\Lambda_1, b] \\ a + r q_{st}\left(I: \frac{y-b}{\Lambda_2}\right), & y \in (b, \Lambda_2+b] \\ a + r + (1-a-r) q_{st}\left(P: \frac{y-b-\Lambda_2}{1-b-\Lambda_2}\right), & y \in (\Lambda_2+b, 1], \end{cases} \tag{72}$$

$$q_{st}(R:y) = \begin{cases} (b-\Lambda_1) q_{st}\left(I: \frac{y}{b-\Lambda_1}\right), & y \in (0, b-\Lambda_1] \\ b-\Lambda_1 + (1-b+\Lambda_1) q_{st}\left(R: \frac{y-b+\Lambda_1}{1-b+\Lambda_1}\right), & y \in (b-\Lambda_1, 1], \end{cases} \tag{73}$$

$$q_{st}(P:y) = \begin{cases} (1-b-\Lambda_2) q_{st}\left(I: \frac{y}{1-b-\Lambda_2}\right), & y \in (0, 1-b-\Lambda_2] \\ 1-b-\Lambda_2 + (b+\Lambda_2) q_{st}\left(P: \frac{y-1+b+\Lambda_2}{b+\Lambda_2}\right), & y \in (1-b-\Lambda_2, 1]. \end{cases} \tag{74}$$

One can show, by exactly the same method as before, that the partially integrated function $Q_t(\alpha:x, y)$ approaches the stationary state $Q_{st}(\alpha:x, y)$ provided that the initial function $Q_0(\alpha:x, y)$ is continuously differentiable with respect to x . Then since the asymptotic measure $Q_{st}(\alpha:x, y)$ is absolutely continuous with respect to the Lebesgue measure along the expanding x direction, it is the SRB measure. Note that, in this case, the quantity $Q_{st}(I)$ is a functional of the initial distribution $Q_0(\alpha:x, y)$.

C. A stationary state under a flux boundary condition

As shown in Sec. V A, one has a flux from the reactant to the product when their concentrations are fixed to the values different from the equilibrium ones. This can be realized in the Baker-type model by fixing the measures of the unit squares corresponding to the reactant and the product to uniform Lebesgue measures with different densities. So we set $Q_t(R:x, y) = R_{ex}y$ and $Q_t(P:x, y) = P_{ex}y$ ($t=0, 1, \dots$). Note

that the same procedure was used to achieve the flux boundary condition for the finite multi-Baker chain.¹³ Then, we have only one time-dependent variable $Q_t(I;x,y)$, which will be abbreviated as $Q_t(I;x,y) \equiv Q_t(x,y)$. Then the equation of motion (67) for the partially integrated distribution function Q_t becomes

$$Q_{t+1}(x,y) = \mathcal{F}' Q_t(x,y) + R_t(x,y) + S(y), \tag{75}$$

where the contraction mapping \mathcal{F}' and $Q_t(x,1)$ -dependent part $R_t(x,y)$ are given by Eqs. (B3) and (B4) of Appendix B, respectively, and the source term $S(y)$ is due to the reactant and product states

$$S(y) = \begin{cases} 0, & y \in [0, \Lambda_1] \\ R_{\text{ex}}(y - \Lambda_1), & y \in (\Lambda_1, b) \\ R_{\text{ex}}(b - \Lambda_1), & y \in [b, \Lambda_2 + b] \\ R_{\text{ex}}(b - \Lambda_1) + P_{\text{ex}}(y - b - \Lambda_2), & y \in (\Lambda_2 + b, 1]. \end{cases} \tag{76}$$

By exactly the same argument as before, Eqs. (75) and (76) are found to have a solution

$$Q_t(x,y) = \frac{b - \Lambda_1}{1 - l - r} R_{\text{ex}} \eta_R(y) + \frac{1 - b - \Lambda_2}{1 - l - r} P_{\text{ex}} \eta_P(y) + \nu^t \int_0^1 dH(x') \left\{ Q_0(x', 1) - \frac{b - \Lambda_1}{1 - l - r} R_{\text{ex}} - \frac{1 - b - \Lambda_2}{1 - l - r} P_{\text{ex}} \right\} G(y) + O(t \lambda^{-t}), \tag{77}$$

where $G(y)$ and $H(x)$ are singular functions introduced in Sec. IV [cf. Eqs. (49) and (50)], and the functions $\eta_R(y)$ and $\eta_P(y)$ are the unique solutions of de Rham equations

$$\eta_R(y) = \begin{cases} l \eta_R\left(\frac{y}{\Lambda_1}\right), & y \in [0, \Lambda_1] \\ 1 - r + \frac{1 - l - r}{b - \Lambda_1} (y - b), & y \in (\Lambda_1, b) \\ r \eta_R\left(\frac{y - b}{\Lambda_2}\right) + 1 - r, & y \in [b, \Lambda_2 + b] \\ 1, & y \in (\Lambda_2 + b, 1] \end{cases} \tag{78}$$

$$\eta_P(y) = \begin{cases} l \eta_P\left(\frac{y}{\Lambda_1}\right), & y \in [0, \Lambda_1] \\ l, & y \in (\Lambda_1, b) \\ r \eta_P\left(\frac{y - b}{\Lambda_2}\right) + l, & y \in [b, \Lambda_2 + b] \\ 1 + \frac{1 - l - r}{1 - b - \Lambda_2} (y - 1), & y \in (\Lambda_2 + b, 1]. \end{cases} \tag{79}$$

Therefore the stationary measure μ_{fl} for this open system is given by

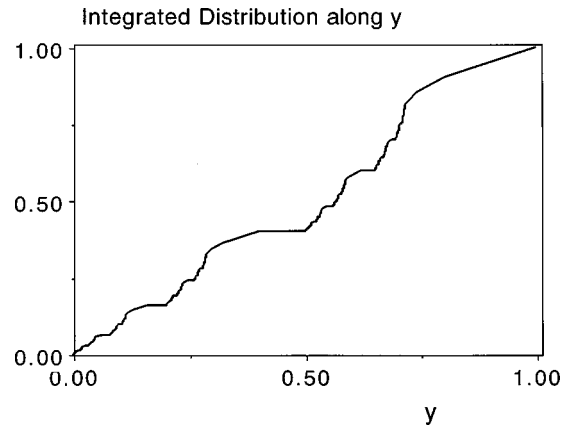


FIG. 10. Partially integrated distribution of the stationary measure μ_{fl} along x for the reaction model under a flux boundary condition. The parameters are chosen as $\Lambda_1=0.4$, $\Lambda_2=0.3$, $b=0.5$, $l=0.4$, $r=0.5$, $R_{\text{ex}}=0$, and $P_{\text{ex}}=(1-l-r)/(1-b-\Lambda_2)$.

$$\mu_{\text{fl}}([0,x] \times [0,y]) = \frac{b - \Lambda_1}{1 - l - r} R_{\text{ex}} x \eta_R(y) + \frac{1 - b - \Lambda_2}{1 - l - r} P_{\text{ex}} x \eta_P(y). \tag{80}$$

As it is smooth (i.e., absolutely continuous with respect to the Lebesgue measure) along the expanding coordinate x , it is an SRB measure. However, the measure μ_{fl} is different from the SRB measures obtained before since it is absolutely continuous with respect to y . It is only weakly singular in the sense that the density is discontinuous on a Cantor-like set (cf., Fig. 10). Actually, such stationary measures become truly singular only for infinite systems.^{13,15} It is interesting to observe that the conditionally invariant measure $\mu_{\text{ph}}([0,x] \times [0,y]) = xG(y)$ appears in the time evolution of the measure (77) as the decay mode, i.e., as the Pollicott-Ruelle resonance.

Now we investigate macroscopic aspects. The total measure $\mu_t((0,1]^2) \equiv \int_0^1 dx Q_t(x,1)$ corresponding to the concentration of the intermediate I is

$$\mu_t((0,1]^2) = \frac{b - \Lambda_1}{1 - l - r} R_{\text{ex}} + \frac{1 - b - \Lambda_2}{1 - l - r} P_{\text{ex}} + \nu^t \delta \mu_0 + O(t \lambda^{-t}), \tag{81}$$

where $\delta \mu_0$ stands for the deviation of the initial measure from the stationary state μ_{fl}

$$\delta \mu_0 \equiv \int_0^1 dH(x') \left\{ Q_0(x', 1) - \frac{b - \Lambda_1}{1 - l - r} R_{\text{ex}} - \frac{1 - b - \Lambda_2}{1 - l - r} P_{\text{ex}} \right\}.$$

The stationary state μ_{fl} admits a flux $J_{R \rightarrow P}$ from the reactant state to the product state. Indeed, the measure $\mu_{\text{fl}}((a+r, 1] \times (0,1])$ is transferred to the product state and the measure $(1-b-\Lambda_2)P_{\text{ex}}$ is transferred from there. This implies the existence of the flux $J_{R \rightarrow P}$ given by

$$\begin{aligned}
 J_{R \rightarrow P} &= \mu_{\text{fl}}((a+r, 1] \times (0, 1]) - (1-b-\Lambda_2)P_{\text{ex}} \\
 &= \frac{(1-a-r)(b-\Lambda_1)}{1-l-r} R_{\text{ex}} \\
 &\quad - \frac{(a-l)(1-b-\Lambda_2)}{1-l-r} P_{\text{ex}}. \tag{82}
 \end{aligned}$$

With the specification (69) of the rate coefficients, the expressions (81) of the stationary measure and the decay mode as well as (82) of the flux agree with their phenomenological counterparts (61), (62) and (63), respectively. In short, the measure μ_{fl} describes a nonequilibrium stationary state which has a nonvanishing flux $J_{R \rightarrow P}$.

VI. CONCLUSIONS

We have explicitly constructed SRB measures for three Baker-type maps employing the similarity between the de Rham equation and the evolution equation of partially integrated distribution functions. Now we give a few more remarks.

- (a) In our examples discussed in Secs. III and IV, we have encountered different types of contracting dynamics that lead to invariant states all of which have singular measures supported on fractal sets. In the first case we considered a nonconservative reversible Baker map on the unit square which globally preserves the area of the square but forms an attractor-repeller pair due to the local contraction and expansion properties of the map. The invariant set is an attractor and is fractal since it has the information dimension less than 2, but it is dense in the unit square. In the second example of the Baker map with a Cantor-like invariant set, we considered two different cases—global contraction onto a fractal set with and without escape of points from the unit square. When there is no escape, the invariant set is a fractal attractor which is a nondense subset of the unit square. If we add the possibility of escape, then the invariant set is fractal in both x and y directions.
- (b) It is worth mentioning that one can convert a physical measure for a system with escape to the proper invariant measure on the repeller by incorporating into the averaging process described in Eq. (5) both the characteristic function on the region of the phase space from which points escape, as well as a ‘‘survival’’ function which is unity if the phase point is in the region for the time interval $0 < \tau < T$, with $T > t$, and zero otherwise. Then by taking the limit $T \rightarrow \infty$, one recovers the invariant measure on the repeller. Such a procedure was used by van Beijeren and Dorfman in order to correctly compute the Lyapunov exponents on the repeller for a Lorentz gas.²⁷ This procedure is closely related to one described by Hunt, Ott, and Yorke²⁸ to obtain natural measures on invariant sets.
- (c) We have encountered three different physical measures, which are characterized by the smoothness along the expanding direction. For closed nonconservative systems, the physical measures are singular and invariant. For an open system under an absorbing boundary

condition, the physical measure is singular and conditionally invariant. And for an open system under a flux boundary condition, the physical measure is invariant, but absolutely continuous with respect to the two-dimensional Lebesgue measure. It is only weakly singular in the sense that the density is discontinuous on a Cantor-like set.

It is interesting to note that the conditionally invariant measures appear as the decay modes (i.e., the Pollicott-Ruelle resonances) for an open system under a flux boundary condition (cf. Sec. V C). Such a relation exists more generally. The Pollicott-Ruelle resonances are defined as the functionals Γ acting on a dynamical variable φ ,^{14,32-38} which satisfy, in case of a map,

$$\langle \Gamma, \varphi \circ \Phi^t \rangle = \zeta^t \langle \Gamma, \varphi \rangle,$$

where Φ is a map and ζ ($|\zeta| < 1$) is a decay rate. When the characteristic function χ_A of any Borel set A is in the domain of the functional Γ and $\langle \Gamma, \chi_M \rangle \neq 0$ with M the whole phase space, then the (possibly complex) measure defined by

$$\mu(A) = \langle \Gamma, \chi_A \rangle,$$

is conditionally invariant since $\mu(\Phi^{-1}A)/\mu(\Phi^{-1}M) = \mu(A)$. Such examples are the hydrodynamic modes for open Hamiltonian systems.^{14,36,37}

- (d) For a nonconservative reversible map Φ , we find different SRB measures μ_{ph} and $\bar{\mu}_{\text{ph}}$ for the forward and backward time evolutions, respectively. For each time evolution, one plays a role of an attracting measure and the other of a repelling measure. This is a key element of the compatibility between dynamical reversibility and irreversible behavior of statistical ensembles. Indeed, when the dynamics is reversible and statistical ensembles irreversibly approach a stationary ensemble μ_{ph} for $t \rightarrow +\infty$, there should exist another stationary ensemble $\bar{\mu}_{\text{ph}}$ which is obtained from μ_{ph} by the time reversal operation. Since the new stationary ensemble $\bar{\mu}_{\text{ph}}$ is repelling, its existence is compatible with the irreversible behavior of statistical ensembles. A similar situation was observed for open Hamiltonian systems under a flux boundary condition.^{13,14} Such systems admit two different stationary states, one is the time reversed state of the other. In this case, an attracting stationary state for the forward time evolution obeys Fick’s law and a repelling state obeys anti-Fick’s law.
- (e) As mentioned in Introduction, an SRB measure for a map may be constructed as follows:³⁰ (1) Approximate the measure by iterating an initial constant density finite times, (2) calculate the average with its result and (3) take the limit of infinite iterations. On the other hand, in our construction of an SRB measure, a functional equation for the partially integrated distribution function similar to the de Rham equation is derived from the evolution equation for measures. Note that the functional equation is a direct consequence of the self-

similarity of the measure. An iterative method of solving the functional equation is similar to the procedure explained above. However, our method has some advantages. First, the de Rham-type functional equations are fixed point equations for contraction mappings and, as a result, the iterative solution strongly converges to the limit. On the contrary, the iterative approximation of the density function does not converge by itself. Therefore, one can obtain a good approximation to the cumulative distribution function of the SRB measure by less iterations than by the conventional method. Second, in the functional equation approach, one can systematically extract exponentially decaying terms which are typically higher order derivatives of singular functions in the sense of Schwartz' distributions. Third, although the measure is defined as the solution of a functional equation, the average values of certain dynamical functions can be calculated analytically as illustrated in Sec. II. So far, the functional equation method was mainly applied to piecewise linear one-dimensional maps and Baker-type maps,^{13,37,38} but we believe that the method can also be applied to other systems if the expanding and contracting directions can be well-separated.

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APPENDIX A: PROOFS OF EQ. (19) AND THE LEMMA

1. Derivation of Eq. (19)

As explained in Sec. III, the evolution equation (18) for the partially averaged distribution function $P_t(x,1) \equiv \int_0^1 dy' \rho_t(x,y')$

$$P_{t+1}(x,1) = rP_t(rx+l,1) + lP_t(lx,1), \tag{A1}$$

is the Frobenius-Perron equation for a one-dimensional exact map $Sx = x/l$ (for $x \in [0,l]$) and $Sx = (x-l)/r$ (for $x \in (l,1]$), which admits the Lebesgue measure as the invari-

ant measure. As a consequence, the integral of $P_t(x,1)$ over the unit interval $[0,1]$ is invariant. Indeed, by integrating Eq. (A1), we have

$$\begin{aligned} \int_0^1 dx P_{t+1}(x,1) &= \int_0^1 dx P_t(x,1) \\ &= \dots = \int_0^1 dx P_0(x,1). \end{aligned}$$

On the other hand, since ρ_0 and, hence, P_0 is continuously differentiable in x , Eq. (A1) implies that $P_t(x,1)$ ($t=1,2,\dots$) are also continuously differentiable in x . And the derivative $\partial_x P_t(x,1)$ satisfies the equation

$$\partial_x P_{t+1}(x,1) = r^2 \partial_x P_t(rx+l,1) + l^2 \partial_x P_t(lx,1),$$

which leads to an inequality

$$\begin{aligned} \|\partial_x P_{t+1}(\cdot,1)\| &\leq (r^2 + l^2) \|\partial_x P_t(\cdot,1)\| \\ &\leq \lambda \|\partial_x P_t(\cdot,1)\| \leq \dots \leq \lambda^{t+1} \|\partial_x P_0(\cdot,1)\|, \end{aligned}$$

where the function norm is defined by $\|\partial_x P_t(\cdot,1)\| \equiv \sup_{x \in [0,1]} |\partial_x P_t(x,1)|$ and $\lambda = \max(l,r) < 1$. Since the function $P_t(x,1)$ can be represented as

$$\begin{aligned} P_t(x,1) - \int_0^1 dx' P_t(x',1) &= \int_0^1 dx' x' \partial_x P_t(x',1) \\ &\quad - \int_x^1 dx' \partial_x P_t(x',1), \end{aligned}$$

we finally obtain

$$\begin{aligned} \left| P_t(x,1) - \int_0^1 dx' P_0(x',1) \right| &\leq \left\{ \int_0^1 dx' x' + \int_x^1 dx' \right\} \|\partial_x P_t(\cdot,1)\| \\ &\leq \frac{3}{2} \|\partial_x P_t(\cdot,1)\| \leq \frac{3}{2} \lambda^t \|\partial_x P_0(\cdot,1)\|, \end{aligned}$$

or Eq. (19).

2. Proof of the lemma

Since the statement of the lemma given in the text is not technically complete, we first give the precise statement and then prove it.

Lemma: Let \mathcal{F} be a linear contraction mapping with the contraction constant $0 < \lambda < 1$ defined on a (Banach) space of bounded functions equipped with the supremum norm $\|\cdot\|$ (i.e., $\|\mathcal{F}f\| \leq \lambda \|f\|$). And let $g_t \equiv g^{(0)} + v'g^{(1)} + g_t^{(2)}$ be a given function where $g^{(0)}$, $g^{(1)}$ and $g_t^{(2)}$ are bounded functions, v is a constant satisfying $\lambda < |v| \leq 1$, and $\|g_t^{(2)}\| \leq K\lambda^t$ with some constant $K > 0$. Then, the solution of the functional equation

$$f_{t+1} = \mathcal{F}f_t + g_t, \tag{A2}$$

is given by

$$f_t = f_\infty^{(0)} + v^t f_\infty^{(1)} + O(t\lambda^t), \tag{A3}$$

where $f_\infty^{(0)}$ and $f_\infty^{(1)}$ are the unique solutions of the following fixed point equations

$$f_\infty^{(0)} = \mathcal{F}f_\infty^{(0)} + g^{(0)}, \tag{A4}$$

$$f_\infty^{(1)} = \frac{1}{\nu} \mathcal{F}f_\infty^{(1)} + \frac{g^{(1)}}{\nu}. \tag{A5}$$

The proof of the lemma is straightforward: From Eq. (A2), one obtains

$$f_t = \mathcal{F}^t f_0 + \sum_{s=0}^{t-1} \mathcal{F}^s g_{t-1-s}.$$

By rewriting the right hand side in terms of $g^{(0)}$, $g^{(1)}$ and $g_t^{(2)}$, this relation leads to

$$f_t = \sum_{s=0}^{\infty} \mathcal{F}^s g^{(0)} + \nu^t \sum_{s=0}^{\infty} \nu^{-s-1} \mathcal{F}^s g^{(1)} + \left\{ \mathcal{F}^t f_0 + \sum_{s=0}^{t-1} \mathcal{F}^s g_{t-1-s}^{(2)} - \sum_{s=t}^{\infty} \mathcal{F}^s g^{(0)} - \nu^t \sum_{s=t}^{\infty} \nu^{-s-1} \mathcal{F}^s g^{(1)} \right\}.$$

The first sum $\sum_{s=0}^{\infty} \mathcal{F}^s g^{(0)}$ in the right-hand side is $f_\infty^{(0)}$ and the second sum $\sum_{s=0}^{\infty} \nu^{-s-1} \mathcal{F}^s g^{(1)}$ is $f_\infty^{(1)}$, which satisfy Eqs. (A4) and (A5), respectively. By repeatedly using the property of \mathcal{F} , we have

$$\|f_t - f_\infty^{(0)} - \nu^t f_\infty^{(1)}\| \leq \lambda^t \left\{ \|f_0\| + \frac{\|g^{(0)}\|}{1-\lambda} + \frac{\|g^{(1)}\|}{|\nu|-\lambda} + Kt \right\},$$

which is $O(t\lambda^t)$ and implies the desired result (A3).

APPENDIX B: PHYSICAL MEASURE FOR A BAKER MAP WITH A CANTOR-LIKE INVARIANT SET

In this Appendix, we outline the derivation of Eq. (48), which is quite similar to that of Eq. (30).

From the definition (47) of the map Ψ , one finds that the Frobenius-Perron equation governing the time evolution of the distribution function $\rho_t(x,y)$ is given by

$$\rho_{t+1}(x,y) = \begin{cases} \frac{l}{\Lambda_1} \rho_t\left(lx, \frac{y}{\Lambda_1}\right), & y \in [0, \Lambda_1] \\ 0, & y \in (\Lambda_1, b) \\ \frac{r}{\Lambda_2} \rho_t\left(rx+a, \frac{y-b}{\Lambda_2}\right), & y \in [b, \Lambda_2+b] \\ 0, & y \in (\Lambda_2+b, 1] \end{cases} \tag{B1}$$

which leads to a contractive time evolution of the partially integrated distribution function $Q_t(x,y) \equiv \int_0^y dy' \rho_t(x,y')$:

$$Q_{t+1}(x,y) = \mathcal{F}' Q_t(x,y) + R_t(x,y), \tag{B2}$$

where \mathcal{F}' stands for a contraction mapping

$$\mathcal{F}' Q_t(x,y) \equiv \begin{cases} l Q_t\left(lx, \frac{y}{\Lambda_1}\right), & y \in [0, \Lambda_1] \\ 0, & y \in (\Lambda_1, b) \\ r Q_t\left(rx+a, \frac{y-b}{\Lambda_2}\right), & y \in [b, \Lambda_2+b] \\ 0, & y \in (\Lambda_2+b, 1] \end{cases} \tag{B3}$$

and $R_t(x,y)$ is a function of $Q_t(x,1)$

$$R_t(x,y) = \begin{cases} 0, & y \in [0, \Lambda_1] \\ l Q_t(lx, 1), & y \in (\Lambda_1, b) \\ l Q_t(lx, 1), & y \in [b, \Lambda_2+b] \\ l Q_t(lx, 1) + r Q_t(rx+a, 1), & y \in (\Lambda_2+b, 1]. \end{cases} \tag{B4}$$

In terms of $Q_t(x,y)$, the expectation value of a dynamical variable $\varphi(x,y)$ is expressed as

$$\int_{[0,1]^2} \varphi(x,y) \rho_t(x,y) dx dy = \int_{[0,1]^2} \varphi(x,y) dx d_y Q_t(x,y), \tag{B5}$$

where d_y stands for the Riemann-Stieltjes integral of Q_t with respect to y .⁵³

To solve Eqs. (B2), (B3) and (B4), we first investigate the equation of motion of $Q_t(x,1)$

$$Q_{t+1}(x,1) = l Q_t(lx, 1) + r Q_t(rx+a, 1). \tag{B6}$$

We observe that, in terms of a function $H(x)$ defined by a de Rham equation (50), one has

$$\begin{aligned} \int_0^1 dH(x') Q_t(x', 1) &= \nu \int_0^1 dH(x') Q_{t-1}(x', 1) \cdots \\ &= \nu^t \int_0^1 dH(x') Q_0(x', 1), \end{aligned} \tag{B7}$$

$$\begin{aligned} Q_t(x,1) - \int_0^1 dH(x') Q_t(x', 1) &= \int_0^1 dx' \{H(x') - \theta(x'-x)\} \partial_{x'} Q_t(x', 1), \end{aligned} \tag{B8}$$

where $\nu = l+r (\leq 1)$ and $\lambda' = \max(l,r) (< 1)$. On the other hand, Eq. (B6) leads to an inequality

$$\|\partial_x Q_t(\cdot, 1)\| \leq \lambda' \|\partial_x Q_{t-1}(\cdot, 1)\| \leq \dots \leq \lambda'^t \|\partial_x Q_0(\cdot, 1)\|,$$

with $\|\partial_x Q_t(\cdot, 1)\| \equiv \sup_{x \in [0,1]} |\partial_x Q_t(x, 1)|$. Then, one obtains from Eqs. (B7) and (B8)

$$Q_t(x,1) = \nu^t \int_0^1 dH(x') Q_0(x', 1) + O(\lambda'^t),$$

and, thus

$$R_t(x,y) = \nu^t \int_0^1 dH(x') Q_0(x', 1) g^{(1)}(y) + O(\lambda'^t), \tag{B9}$$

$$g^{(1)}(y) = \begin{cases} 0, & y \in [0, \Lambda_1] \\ l, & y \in (\Lambda_1, \Lambda_2+b] \\ l+r, & y \in (\Lambda_2+b, 1]. \end{cases} \tag{B10}$$

Since $\nu = l+r > \max(l,r) = \lambda'$, the lemma of Sec. III can be applied to the evolution equations (B2), (B3) and (B4) and one obtains

$$Q_t(x,y) = \nu^t \int_0^1 dH(x') Q_0(x', 1) G(y) + O(t\lambda'^t). \tag{B11}$$

where $G(y)$ is the solution of a de Rham equation (49). The desired result (48) immediately follows from Eq. (B11).

APPENDIX C: ON AN INVARIANT MEASURE μ_{in} FOR A BAKER MAP WITH ESCAPE

In this Appendix, we show that the invariant measure μ_{in} on the fractal repeller considered in Sec. IV is a Gibbs measure.

To show that, we first observe that the image $\Psi^m[0,1]^2$ of the unit square by the map Ψ^m consists of 2^m horizontal strips, which will be referred to as $H_1^{[m]}, H_2^{[m]}, \dots, H_{2^m}^{[m]}$; and that the preimage of $\Psi^{-n}[0,1]^2$ of the unit square by the map Ψ^n consists of 2^n vertical strips, which will be referred to as $V_1^{[n]}, V_2^{[n]}, \dots, V_{2^n}^{[n]}$. Then, boxes $H_i^{[m]} \cap V_j^{[n]}$ generated by the overlap procedure provide a generating partition of the repeller. As easily seen, for each box $H_i^{[m]} \cap V_j^{[n]}$, a stretching factor for a time interval $[-m, n-1]$

$$u_{ij}(n, m) \equiv \sum_{t=-m}^{n-1} \lambda_x(\Psi^t(x, y)), \tag{C1}$$

does not depend on the initial point $(x, y) \in H_i^{[m]} \cap V_j^{[n]}$, where $\lambda_x(x, y)$ is the local expanding rate defined by $\lambda_x(x, y) = -\ln l$ (if $x \in [0, l]$) and $\lambda_x(x, y) = -\ln r$ (if $x \in [a, a+r]$). Note that, when $(x, y) \in H_i^{[m]} \cap V_j^{[n]}$, the preimage $\Psi^{-k}(x, y)$ is unique for $k = 1, 2, \dots, m$. We show that the μ_{in} -measure of the box $H_i^{[m]} \cap V_j^{[n]}$ is given by

$$\mu_{in}(H_i^{[m]} \cap V_j^{[n]}) = \frac{e^{-u_{ij}(n, m)}}{\sum_{i,j} e^{-u_{ij}(n, m)}}, \tag{C2}$$

which implies that the measure μ_{in} is a Gibbs measure.^{1,18,19,24} Note that, since the numerator of Eq. (C2) is a product $l^s r^{n+m-s}$ of $(n+m)$ factors ($s = 0, 1, 2, \dots, n+m$) and there are $(n+m)! / \{s!(n+m-s)!\}$ boxes with this value, the sum of the numerators, or the normalization factor, is

$$\begin{aligned} \sum_{i,j} e^{-u_{ij}(n, m)} &= \sum_{s=0}^{n+m} \frac{(n+m)!}{s!(n+m-s)!} l^s r^{n+m-s} \\ &= (l+r)^{n+m} = e^{-(n+m)\kappa}, \end{aligned}$$

where $\kappa = -\ln(l+r)$ is the escape rate.²⁴

Before proving Eq. (C2), we verify it for a simple case $n=2$ and $m=1$. In this case, we have $V_1^{[2]} = [0, l^2] \times [0, 1]$, $V_2^{[2]} = [a, r l + a] \times [0, 1]$, $V_3^{[2]} = [l a, l r + l a] \times [0, 1]$, $V_4^{[2]} = [r a + a, r^2 + r a + a] \times [0, 1]$; $H_1^{[1]} = [0, 1] \times [0, \Lambda_1]$ and $H_2^{[1]} = [0, 1] \times [b, \Lambda_2 + b]$. As an example, we consider a box $H_2^{[1]} \cap V_3^{[2]} = [l a, l r + l a] \times [b, \Lambda_2 + b]$. For any point $(x, y) \in H_2^{[1]} \cap V_3^{[2]}$, we have $0 \leq x \leq l$ and $\Psi(x, y), \Psi^{-1}(x, y) \in [a, a+r] \times [0, 1]$ and, thus

$$u_{2,3}(2, 1) \equiv \sum_{t=-1}^1 \lambda_x(\Psi^t(x, y)) = -2 \ln r - \ln l.$$

On the other hand, from the functional equations of $H(x)$ and $G(x)$, we have

$$\begin{aligned} \mu_{in}(H_2^{[1]} \cap V_3^{[2]}) &= \{H(lr+la) - H(la)\} \{G(\Lambda_2+b) - G(b)\} \\ &= \frac{l}{l+r} \frac{r}{l+r} \{H(1) - H(0)\} \frac{r}{l+r} \{G(1) - G(0)\} \\ &= \frac{lr^2}{(l+r)^3} = \frac{\exp(-u_{2,3}(2,1))}{(l+r)^3}, \end{aligned}$$

which is (C2).

Now we go to the proof of Eq. (C2). Since one can write $H_i^{[m]} = [0, 1] \times [\alpha_i^{[m]}, \beta_i^{[m]}]$ and $V_j^{[n]} = [\gamma_j^{[n]}, \delta_j^{[n]}] \times [0, 1]$, and thus

$$\begin{aligned} \mu_{in}(H_i^{[m]} \cap V_j^{[n]}) &= \{G(\beta_i^{[m]}) - G(\alpha_i^{[m]})\} \{H(\delta_j^{[n]}) - H(\gamma_j^{[n]})\} \\ &= \mu_{in}(H_i^{[m]}) \mu_{in}(V_j^{[n]}), \end{aligned}$$

it is enough to show

$$\mu_{in}(V_j^{[n]}) = \frac{e^{-u_j(n)}}{(l+r)^n}, \tag{C3}$$

and

$$\mu_{in}(H_i^{[m]}) = \frac{e^{-u_i(m)}}{(l+r)^m}, \tag{C4}$$

where $u_j(n) = \sum_{t=0}^{n-1} \lambda_x(\Psi^t(x, y))$ and $u_i(m) = \sum_{t=-1}^{-m} \lambda_x(\Psi^t(x, y))$ are stretching factors for a vertical strip $V_j^{[n]}$ and a horizontal strip $H_i^{[m]}$, respectively. As before, the stretching factors are constant on each strip. The relations (C3) and (C4) are proved by induction with respect to n and m , with the aid of the functional equations for $H(x)$ and $G(y)$, respectively. Since the proofs of the two relations are similar, we show (C3) only.

It is easy to see that (C3) holds for $n=1$. Now we suppose that Eq. (C3) is valid for n . As easily seen, a vertical strip $V_{j'}^{[n+1]}$ is expressed by some vertical strip $V_j^{[n]}$ as

$$V_{j'}^{[n+1]} = R_\sigma \cap \Psi^{-1} V_j^{[n]},$$

where $\sigma = 0$ or 1 with $R_0 = [0, l] \times [0, 1]$ and $R_1 = [a, a+r] \times [0, 1]$. In terms of $\gamma_j^{[n]}$ and $\delta_j^{[n]}$, we have

$$V_{j'}^{[n+1]} = \begin{cases} [l \gamma_j^{[n]}, l \delta_j^{[n]}] \times [0, 1], & \sigma = 0 \\ [r \gamma_j^{[n]} + a, r \delta_j^{[n]} + a] \times [0, 1], & \sigma = 1. \end{cases}$$

When $\sigma = 0$, from the functional equation for $H(x)$, one obtains

$$\begin{aligned} \mu_{in}(V_{j'}^{[n+1]}) &= H(l \delta_j^{[n]}) - H(l \gamma_j^{[n]}) \\ &= \frac{l}{l+r} \{H(\delta_j^{[n]}) - H(\gamma_j^{[n]})\} \\ &= \frac{l}{l+r} \mu_{in}(V_j^{[n]}) = \frac{\exp(-u_j(n) + \ln l)}{(l+r)^{n+1}}. \end{aligned} \tag{C5}$$

Then, for $(x, y) \in V_{j'}^{[n+1]}$, one has $(x, y) \in R_0$ and $\Psi(x, y) \in V_j^{[n]}$. Therefore, $\lambda_x(x, y) = -\ln l$ and

$$\begin{aligned}
 u_j(n) - \ln l &= \sum_{i=0}^{n-1} \lambda_x(\Psi^i \circ \Psi(x, y)) + \lambda_x(x, y) \\
 &= \sum_{i=0}^n \lambda_x(\Psi^i(x, y)) = u_{j'}(n+1). \tag{C6}
 \end{aligned}$$

Similarly, one can verify Eq. (C6) when $\sigma = 1$. Hence,

$$\mu_{\text{in}}(V_{j'}^{[n+1]}) = \frac{\exp(-u_{j'}(n+1))}{(l+r)^{n+1}},$$

or Eq. (C3) holds for $n+1$ and, by induction, it is valid for all positive integer n .

APPENDIX D: THE EVOLUTION EQUATION OF MEASURES FOR A REACTION MODEL

In this Appendix, we write down the evolution equation of the partially integrated distribution function $Q_t(\alpha : x, y)$

$= \int_0^y dy' \rho_t(\alpha : x, y')$ ($\alpha = R, I$, or P) for a chemical reaction model introduced in Sec. V. The density function $\rho_t(\alpha : x, y)$ ($\alpha = R, I$, or P) at time t is given by

$$\begin{aligned}
 \rho_t(\alpha : x, y) &= \sum_{\beta=R,I,P} \int_{[0,1]^2} dx dy \delta((\alpha : x, y) \\
 &\quad - \Psi'^t(\beta : x', y')) \rho_0(\beta : x', y'),
 \end{aligned}$$

where ρ_0 is the density function of the initial measure, Ψ' is the map introduced in Sec. V [cf. Eqs. (64), (65), and (66)] and the delta function $\delta((\alpha : x, y) - (\beta : x', y'))$ stands for the product $\delta_{\alpha,\beta} \delta(x-x') \delta(y-y')$.

By integrating the Frobenius-Perron equation for the density $\rho_t(\alpha : x, y)$, one obtains the evolution equation for Q_t

$$Q_{t+1}(\alpha : x, y) = \bar{\mathcal{F}} Q_t(\alpha : x, y) + \bar{R}_t(\alpha : x, y), \tag{D1}$$

where $\alpha = R, I$, or P , a linear contraction mapping $\bar{\mathcal{F}}$ is defined by

$$\bar{\mathcal{F}} Q_t(I : x, y) = \begin{cases} l Q_t\left(I : lx, \frac{y}{\Lambda_1}\right), & y \in (0, \Lambda_1] \\ (b - \Lambda_1) Q_t\left(R : (b - \Lambda_1)x, \frac{y - \Lambda_1}{b - \Lambda_1}\right), & y \in (\Lambda_1, b] \\ r Q_t\left(I : rx + a, \frac{y - b}{\Lambda_2}\right), & y \in (b, \Lambda_2 + b] \\ (1 - b - \Lambda_2) Q_t\left(P : (1 - b - \Lambda_2)x, \frac{y - b - \Lambda_2}{1 - b - \Lambda_2}\right), & y \in (\Lambda_2 + b, 1] \end{cases} \tag{D2}$$

$$\bar{\mathcal{F}} Q_t(R : x, y) = \begin{cases} (a - l) Q_t\left(I : (a - l)x + l, \frac{y}{b - \Lambda_1}\right), & y \in (0, b - \Lambda_1] \\ (1 - b + \Lambda_1) Q_t\left(R : (1 - b + \Lambda_1)x + b - \Lambda_1, \frac{y - b + \Lambda_1}{1 - b + \Lambda_1}\right), & y \in (b - \Lambda_1, 1], \end{cases} \tag{D3}$$

$$\bar{\mathcal{F}} Q_t(P : x, y) = \begin{cases} (1 - a - r) Q_t\left(I : (1 - a - r)x + a + r, \frac{y}{1 - b - \Lambda_2}\right), & y \in (0, 1 - b - \Lambda_2] \\ (b + \Lambda_2) Q_t\left(P : (b + \Lambda_2)x + 1 - b - \Lambda_2, \frac{y - 1 + b + \Lambda_2}{b + \Lambda_2}\right), & y \in (1 - b - \Lambda_2, 1], \end{cases} \tag{D4}$$

and $\bar{R}_t(\alpha : x, y)$ is a functional of $Q_t(\alpha : x, 1)$:

$$\bar{R}_t(I : x, y) = \begin{cases} 0, & y \in (0, \Lambda_1] \\ l Q_t(I : lx, 1), & y \in (\Lambda_1, b] \\ l Q_t(I : lx, 1) + (b - \Lambda_1) Q_t(R : (b - \Lambda_1)x, 1), & y \in (b, \Lambda_2 + b] \\ l Q_t(I : lx, 1) + (b - \Lambda_1) Q_t(R : (b - \Lambda_1)x, 1) + r Q_t(I : rx + a, 1), & y \in (\Lambda_2 + b, 1] \end{cases} \tag{D5}$$

$$\bar{R}_t(R : x, y) = \begin{cases} 0, & y \in (0, b - \Lambda_1] \\ (a - l) Q_t(I : (a - l)x + l, 1), & y \in (b - \Lambda_1, 1], \end{cases} \tag{D6}$$

$$\bar{R}_t(P : x, y) = \begin{cases} 0, & y \in (0, 1 - b - \Lambda_2] \\ (1 - a - r) Q_t(I : (1 - a - r)x + a + r, 1) & y \in (1 - b - \Lambda_2, 1]. \end{cases} \tag{D7}$$

These are the desired results. Note that the contraction constant $\bar{\lambda}$ of the mapping $\bar{\mathcal{F}}$ is given by

$$\bar{\lambda} = \max(l, r, a - l, 1 - a - r, b - \Lambda_1, b + \Lambda_2, 1 - b + \Lambda_1, 1 - b - \Lambda_2) (< 1).$$

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- $$\int_0^1 \varphi(x, y) d_y P_k(x, y) = \lim_{\max(y_k - y_{k-1}) \rightarrow 0} \sum_{k=1}^n \varphi(x, \xi_k) \{P_k(x, y_k) - P_k(x, y_{k-1})\}.$$
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