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Statistically preserved structures and anomalous scaling in turbulent active scalar advection

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Abstract. – The anomalous scaling of correlation functions in the turbulent statistics of active scalars (like temperature in turbulent convection) is understood in terms of an auxiliary passive scalar which is advected by the same turbulent velocity field. The even-order correlation functions of the two fields are the same to leading order (up to a trivial multiplicative factor). The leading correlation functions are statistically preserved structures of the passive scalar decaying problem; thus the universality of the scaling exponents of the even-order correlations of the active scalar is demonstrated.

The understanding of the old riddle of anomalous scaling in *generic* turbulent statistics had advanced recently, albeit in the modest context of passive scalar and passive vector advection [1–3]. Motivated by some recent numerical indications [4], we propose in this letter that under generic conditions a similar understanding can be extended to *active* scalar (or vector) advection by a generic turbulent velocity field. The difference between the two problems is that passive fields leave the velocity statistics intact, satisfying a purely linear problem into which the velocity field is injected as an advection term, while active fields influence the statistics of the velocity field. This development brings us therefore closer to understanding anomalous scaling in the nonlinear velocity problem itself.

Passive fields are advected in turbulence with typical dynamics of the form

$$\frac{\partial \phi(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t)\cdot\boldsymbol{\nabla}\phi(\boldsymbol{r},t) = \kappa\nabla^2\phi(\boldsymbol{r},t) + f(\boldsymbol{r},t). \tag{1}$$

Here κ is the diffusivity and $f(\boldsymbol{r},t)$ is a white random force of zero mean with compact support in $\boldsymbol{k}$-space, acting on the largest scales of the order of the outer scale L only. The advecting velocity field is a generic turbulent field at high Reynolds number with an extended scaling range, typically exhibiting anomalous scaling. We use the word "generic" here to distinguish from model velocity fields with δ-function temporal correlations [5]. Below we will take the velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ that advects the passive scalar from a set of coupled equations of an active scalar field. "Anomalous scaling" means that multi-point correlation functions are homogeneous functions of their arguments, with exponents that cannot be guessed from dimensional

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analysis. Thus, for example, the field $\phi(\boldsymbol{r},t)$ has simultaneous multi-point correlation functions

$$F^{(n)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_n)\equiv\langle\phi(\boldsymbol{r}_1,t)\phi(\boldsymbol{r}_2,t)\cdots\phi(\boldsymbol{r}_n,t)\rangle_f, \tag{2}$$

where pointed brackets with subscript f refer to averaging over the statistics of the advecting velocity field *and* of the forcing. Anomalous scaling means that

$$F^{(n)}(\lambda\boldsymbol{r}_1,\ldots,\lambda\boldsymbol{r}_n)=\lambda^{\zeta_n}F^{(n)}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n), \tag{3}$$

with ζ_n having a non-trivial dependence on n.

In recent work [2,3] it was clarified why and how passive fields exhibit anomalous scaling. The key is to consider a problem associated with eq. (1) which is the *decaying problem* in which the forcing $f(\boldsymbol{r},t)$ is put to zero. The problem becomes then a linear initial-value problem, $\partial\phi/\partial t=\mathcal{L}\phi$, with a formal solution

$$\phi(\boldsymbol{r},t)=\int \mathrm{d}\boldsymbol{r}'\boldsymbol{R}(\boldsymbol{r},\boldsymbol{r}',t)\phi(\boldsymbol{r}',0), \tag{4}$$

with the operator $\boldsymbol{R}\equiv T^+\exp[\int_0^t \mathrm{d}s\mathcal{L}(s)]$, and T^+ being the time ordering operator. Define next the *time-dependent* correlation functions of the decaying problem:

$$C^{(n)}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n,t)\equiv\langle\phi(\boldsymbol{r}_1,t)\cdots\phi(\boldsymbol{r}_n,t)\rangle. \tag{5}$$

Here pointed brackets without subscript f refer to the decaying object in which averaging is taken with respect to realizations of the velocity field only. As a result of eq. (4), the decaying correlation functions are developed by a propagator $\mathcal{P}^{(n)}_{\underline{\boldsymbol{r}}|\underline{\boldsymbol{\rho}}}$ (with $\underline{\boldsymbol{r}}\equiv\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_n$):

$$C^{(n)}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n,t)=\int \mathrm{d}\underline{\boldsymbol{\rho}}\mathcal{P}^{(n)}_{\underline{\boldsymbol{r}}|\underline{\boldsymbol{\rho}}}(t)C^{(n)}(\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_n,0). \tag{6}$$

In writing this equation we made explicit use of the fact that the *initial* distribution of the passive field $\phi(\boldsymbol{r},0)$ is statistically independent of the advecting velocity field. Thus the operator $\mathcal{P}^{(n)}_{\underline{\boldsymbol{r}}|\underline{\boldsymbol{\rho}}}$ can be written explicitly

$$\mathcal{P}^{(n)}_{\underline{\boldsymbol{r}}|\underline{\boldsymbol{\rho}}}(t)\equiv\langle\boldsymbol{R}(\boldsymbol{r}_1,\boldsymbol{\rho}_1,t)\boldsymbol{R}(\boldsymbol{r}_2,\boldsymbol{\rho}_2,t)\cdots\boldsymbol{R}(\boldsymbol{r}_n,\boldsymbol{\rho}_n,t)\rangle. \tag{7}$$

The key finding [2,3] is that the operator $\mathcal{P}^{(n)}_{\underline{\boldsymbol{r}}|\underline{\boldsymbol{\rho}}}$ possesses *left* eigenfunctions of eigenvalue 1, *i.e.* there exist functions $Z^{(n)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_n)$ satisfying

$$Z^{(n)}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n)=\int \mathrm{d}\underline{\boldsymbol{\rho}}\mathcal{P}^{(n)}_{\underline{\boldsymbol{r}}|\underline{\boldsymbol{\rho}}}(t)Z^{(n)}(\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_n). \tag{8}$$

The functions $Z^{(n)}$ are referred to as "statistically preserved structures", being invariant to the dynamics, even though *the operator is strongly time dependent and decaying.* How to form, from these functions, infinitely many conserved variables in the decaying problem was shown in [2]. The functions $Z^{(n)}(\underline{\boldsymbol{r}})$ are homogeneous functions of their arguments, with anomalous scaling exponents ζ_n. More importantly, it was shown that the correlation functions of the forced case, $F^{(n)}(\underline{\boldsymbol{r}})$, have exactly the same scaling exponents as $Z^{(n)}(\underline{\boldsymbol{r}})$ [3]. In the scaling sense

$$F^{(n)}(\underline{\boldsymbol{r}})\sim Z^{(n)}(\underline{\boldsymbol{r}}). \tag{9}$$

This is how anomalous scaling in passive fields is understood.

Consider next the dynamics of an active field. For concreteness we will think about temperature in turbulent Boussinesq convection, but the ideas are immediately generalized to other types of active fields. The equation of motion is formally identical to eq. (1):

$$\frac{\partial T(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t)\cdot\boldsymbol{\nabla}T(\boldsymbol{r},t) = \kappa\nabla^2 T(\boldsymbol{r},t) + f(\boldsymbol{r},t), \tag{10}$$

but now the velocity field is affected by the temperature. For an incompressible fluid of unit density [6],

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}p + \nu\nabla^2\boldsymbol{u} + \alpha g T\hat{z}. \tag{11}$$

Here p, ν, α, g and $\hat{z}$ are the pressure, kinematic viscosity, thermal expansion coefficient, acceleration due to gravity and a unit vector in the upward direction, respectively. The appearance of T in the equation for $\boldsymbol{u}$ is crucial, and changes the scaling exponents of $\boldsymbol{u}$ from Kolmogorov (approximately) to Bolgiano (approximately) [6]. It makes no sense now to consider the decaying problem for T; as this field decays, the statistics of the velocity field changes, and there is very little to say. On the other hand, we can learn a great deal from considering the forced solutions, comparing the forced correlation functions of the active field with those of the passive field equation (1) when advected by the same velocity field that satisfies (11). Indeed in [4] it was found that a reduced form of the 4th-order correlation function of both fields appeared identical. Here we opt to consider a shell model that has the same conservation laws and the same form of coupling as eqs. (10) and (11) to investigate this issue in much better numerical detail. We consider a variant of the shell model studied in ref. [7]:

$$\begin{aligned}\frac{\partial T_n}{\partial t} &= \tilde{a}k_n(u_{n-1}T_{n-1} - hu_nT_{n+1}) + \\ &\quad +\tilde{b}k_n(u_nT_{n-1} - hu_{n+1}T_{n+1}) - \kappa k_n^2 T_n + f\delta_{n,0}, \end{aligned}\tag{12}$$

$$\begin{aligned}\frac{\partial u_n}{\partial t} &= ak_n(u_{n-1}^2 - hu_nu_{n+1}) + \\ &\quad +bk_n(u_nu_{n-1} - hu_{n+1}^2) - \nu k_n^2 u_n + T_n, \end{aligned}\tag{13}$$

$$\begin{aligned}\frac{\partial C_n}{\partial t} &= \tilde{a}k_n(u_{n-1}C_{n-1} - hu_nC_{n+1}) + \\ &\quad +\tilde{b}k_n(u_nC_{n-1} - hu_{n+1}C_{n+1}) - \kappa k_n^2 C_n + f\delta_{n,0}. \end{aligned}\tag{14}$$

In this model n stands for the index of a shell of wave vector $k_n = k_0h^n$, with $n = 0, 1, \ldots$, $N-1$. We take $h = 2$, and the parameters used in the simulation are $a = 0.01$, $\tilde{a} = \tilde{b} = b = 1$, $k_0 = 1$, $\kappa = \nu = 5\times10^{-4}$. The number of shells is $N = 30$. The forcing appearing in eqs. (12) and (14) are of the same type, *i.e.* white, Gaussian of zero mean, but we use *different realizations* in the two equations.

Without the coupling to T_n, the velocity equation has an inviscid unstable Kolmogorov fixed point, $u_n \sim k_n^{-1/3}$. This is changed by the coupling [7], and the system of equations for T_n and u_n exhibits an inviscid unstable Bolgiano fixed point, $u_n \sim k_n^{-3/5}$, $T_n \sim k_n^{-1/5}$. The chaotic dynamics renders the statistics of the velocity field strongly non-Gaussian (cf. inset in fig. 1). The exponents for the active scalar are markedly anomalous, whereas, for the velocity, they appear closer to normal.

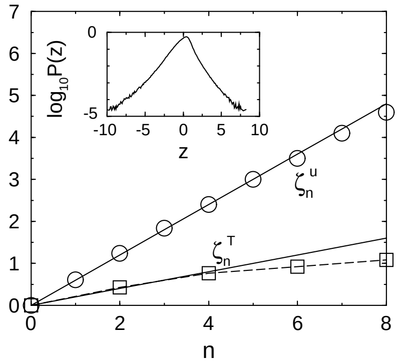

Fig. 1 – The scaling exponents for the velocity field (circles) and the active scalar field (squares). The solid lines are, respectively, $3n/5$ and $n/5$ for the velocity and the active scalar fields. Shown in the inset is the pdf of $z = u_n/\langle u_n^2\rangle^{1/2}$ at shell $n = 14$.

Both the passive and the active forced equations can be solved in the same way; writing the equation as $\dot{T}_n = \mathcal{L}_{n,n'}T_{n'} + f\delta_{n,0}$, the solution is

$$T_n(t) = \sum_{n'} \boldsymbol{R}_{n,n'}(t \mid 0)T_{n'}(0) + \int_0^t \mathrm{d}\tau \boldsymbol{R}_{n,0}(t \mid \tau)f(\tau), \tag{15}$$

with a similar expression for the passive scalar. Here $\boldsymbol{R}_{n,n'}(t \mid \tau) \equiv T^+ \exp[\int_\tau^t \mathrm{d}s\mathcal{L}(s)]_{n,n'}$. For $t \to \infty$, the first term decays to zero, but the second term exists. Next, we encounter the difference between the active and passive case. Computing the average of T, or any other odd moment, we get a finite result:

$$\langle T_n(t)\rangle = \int_0^t \mathrm{d}\tau \langle \boldsymbol{R}_{n,0}(t \mid \tau)f(\tau)\rangle. \tag{16}$$

We cannot decorrelate the random forcing f from the operator $\boldsymbol{R}$ which involves the velocity field. The forcing f is correlated with T, which is itself correlated with $\boldsymbol{u}$ via eq. (11), and therefore the field T has a first and higher odd moments: the probability density function (pdf) of T will be asymmetric. For the passive case (advected by the same velocity field $\boldsymbol{u}$ of eq. (13)), an equation equivalent to eq. (16) can be simplified by decorrelating f from $\boldsymbol{R}$. As the odd moments of f vanish, the passive scalar has zero odd moments, and its pdf is symmetric. We show in fig. 2 the pdf's of $x = \phi_n/\langle\phi_n^2\rangle^{1/2}$, where ϕ is T_n or C_n, for $n = 14$. One clearly sees the symmetry of the pdf of the passive scalar, in contradistinction to the asymmetry of the pdf of the active scalar. This is typical to all n in the inertial range.

The situation is altogether different for the statistics of even moments. To demonstrate the difference we plot in fig. 3 the (typical) pdf of $\tilde{T}_n^2$ and C_n^2 for $n = 9$ and 14, where $\tilde{T}_n \equiv T_n - \langle T_n\rangle$. In plotting we realize that the passive scalar is defined up to a constant, so for the passive scalar the pdf is plotted for the rescaled variable βC_n^2, where $\beta = \langle\tilde{T}_n^2\rangle/\langle C_n^2\rangle \approx 0.6327$. Note that there is only one numerical freedom β, constant for all n in the inertial range. We find very close agreement of all the pdf's in the inertial range. The identity of the pdf's of $\tilde{T}_n^2$ and C_n^2 translates automatically to the identity of the even-order structure functions $S^{(2m)}(k_n) \equiv \langle\tilde{\phi}_n^{2m}\rangle$, where $\tilde{\phi}_n = \tilde{T}_n$ or C_n (up to a constant β^m). This is demonstrated in fig. 4. We see that the 2nd-, 4th- and 6th-order structure functions are barely distinguishable, with

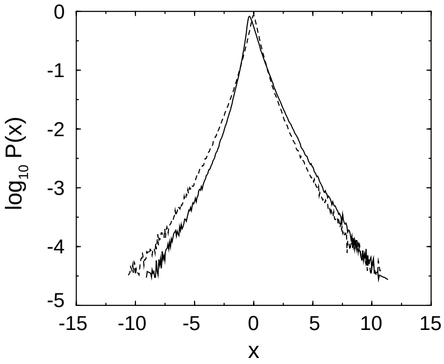

Fig. 2

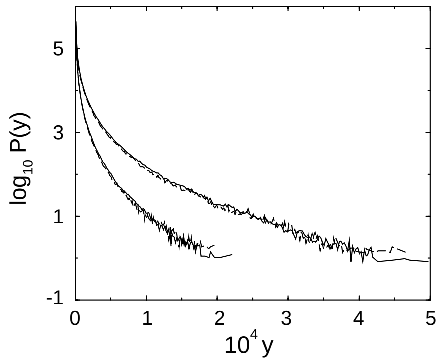

Fig. 3

Fig. 2 – The pdf's of the active (solid) and passive (dashed) scalars at shell $n = 14$. Note that the pdf of the active scalar is asymmetric.

Fig. 3 – The pdf's of y where $y = \tilde{T}_n^2$ (solid) or βC_n^2 (dashed) at shells $n = 9$ and 14.

the same scaling exponents in the inertial range. Finally, we demonstrate that the identity of the statistics of the squares of the passive and active scalars transcends structure functions. Consider for example the multi-point correlation functions $\langle \tilde{T}_n^2 \tilde{T}_{n+5}^2 \rangle$ and $\langle \tilde{T}_n^2 \tilde{T}_{n+5}^2 \tilde{T}_{2n}^2 \rangle$. In fig. 5 these correlation functions are compared to their passive counterparts. The conclusion is that again the multi-point correlation functions are indistinguishable once the passive ones are rescaled by β^q, where q is the overall order of the correlation function. We thus propose that as far as the even-order statistical objects are concerned, the active ones land on the scaling solutions that characterize the statistically preserved structures of the passive counterparts.

To have an intuition about how the active field lands on the scaling solutions of the passive field, consider what happens to any initial condition of the decaying passive problem —see

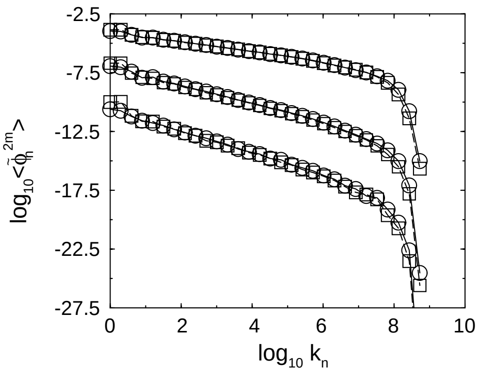

Fig. 4

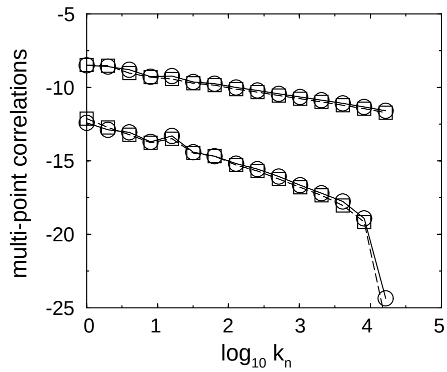

Fig. 5

Fig. 4 – The even-order structure functions $\langle \tilde{T}_n^{2m} \rangle$ (circles) and $\langle \beta^m C_n^{2m} \rangle$ (squares), with $m = 1$, 2 and 3, from top to bottom.

Fig. 5 – Upper curve: $\log_{10}\langle \tilde{T}_n^2 \tilde{T}_{n+5}^2 \rangle$ (circles) and $\log_{10}\langle \beta^2 C_n^2 C_{n+5}^2 \rangle$ (squares). Lower curve: $\log_{10}\langle \tilde{T}_n^2 \tilde{T}_{n+5}^2 \tilde{T}_{2n}^2 \rangle$ (circles) and $\log_{10}\langle \beta^3 C_n^2 C_{n+5}^2 C_{2n}^2 \rangle$ (squares).

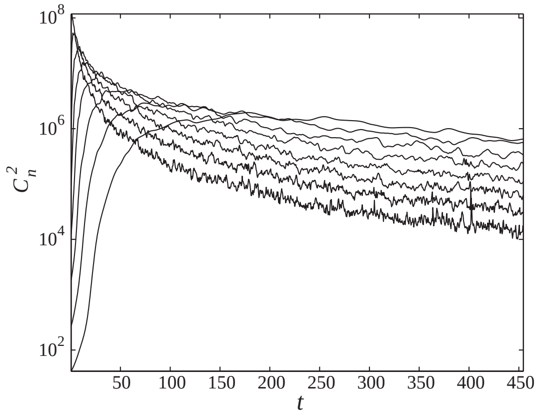

Fig. 6 – An example of the fate of an initial-value term as a function of time, in units of τ_0. Shown are shells $n = 5, 7, \ldots, 19$. The order is $n = 19$ highest at short times, but $n = 5$ highest at long times.

fig. 6. Here we start, as an example, from the initial value $C_n(t = 0) \propto k_n^{2/3}$, such that the order of the amplitudes is inverted with respect to the spectrum of the passive scalar. We plot, as a function of time, the trajectories of $C_n^2(t)$ as computed from this initial condition, averaged over 650 realizations. We see that the trajectories turn around and land *very quickly* on a decaying scaling solution in which the order of the amplitudes and the ratios between them are identical to the spectrum of the zero mode of the passive field; the decay that we see, at a rate proportional to $1/t^2$, is entirely due to dissipative effects, as explained in some detail in [3].

The second-order correlation function $\langle T_n^2 \rangle_f$ of the active scalar, at long times, is nothing but

$$\langle T_n^2 \rangle_f = \int_{-\infty}^{t} \int_{-\infty}^{t} \mathrm{d}\tau \, \mathrm{d}s \langle \boldsymbol{R}_{n,0}(t \mid \tau) \boldsymbol{R}_{n,0}(t \mid s) f(\tau) f(s) \rangle. \tag{17}$$

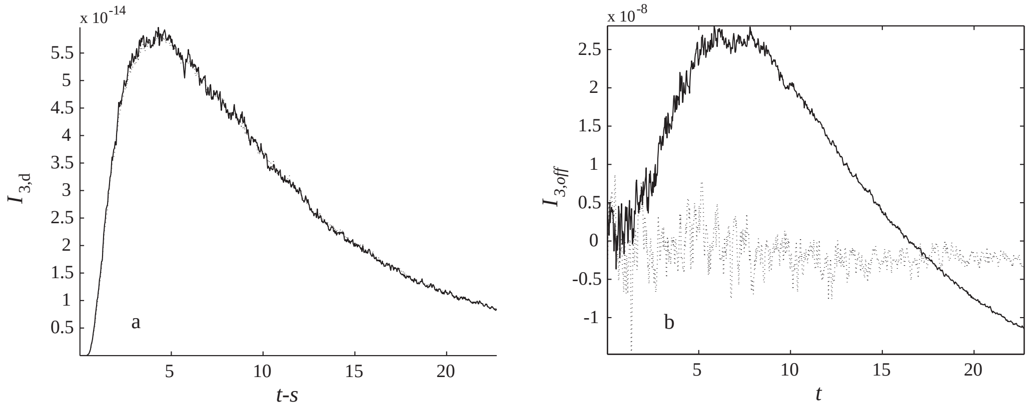

Fig. 7 – Panel a: Comparison of the "diagonal" integrands of the passive and active cases for $n = 10$. The plots are indistinguishable. Panel b: The off-diagonal integrals for $n = 10$, for the passive and active fields, respectively. For the passive field (dotted line) it fluctuates around zero, while for the active field it begins positive, and then turns negative. For longer times it saturates at a constant negative value, giving rise to the factor β.

Each factor $\boldsymbol{R}_{n,0}(t \mid \tau) f(\tau)$ behaves for every τ like the decay problem shown in fig. 6, with the "initial value" $C_n(\tau) = f(\tau)$. For most of the time integration the n-dependence will be dominated by the scaling exponent of the passive statistically preserved structure. In more detail, the double integral has a "diagonal" part $I_{n,d}$ (in which $\tau = s$), $I_{n,d} = \int_{-\infty}^{t} \mathrm{d}\tau \langle \boldsymbol{R}_{n,0}(t \mid \tau) \cdot \boldsymbol{R}_{n,0}(t \mid \tau) f(\tau) f(\tau) \rangle$, and an off-diagonal part which is all the rest of the double integral. For the passive field there exists only a "diagonal" part due to the statistical decorrelation, *i.e.* $\int_{-\infty}^{t} \mathrm{d}\tau \langle \boldsymbol{R}_{n,0}(t \mid \tau) \boldsymbol{R}_{n,0}(t \mid \tau) \rangle f^2$. In fig. 7, panel a, we show the integrands of diagonal parts of the integrals for both the active and the passive fields, for the representative case of $n = 10$. We see that they are indistinguishable, meaning also that the relative amplitudes for different n values agree. Our numerics show that also the off-diagonal parts of the integrals have relative n amplitudes which are determined by the zero mode of the passive field. However, as shown in panel b of fig. 7, the off-diagonal part of the active integral settles for a long time on a *negative* value which is responsible for the factor β. More details on the identity of the active and passive even scaling exponents will be provided in a forthcoming publication [8].

In summary, we offered detailed numerical evidence and a short discussion for a preliminary support of a conjecture that under generic conditions the even-order correlation functions of active scalars can be understood via the emerging theory of statistically preserved structures of the passive scalar counterpart. We expect that the present findings will hold whenever the dynamical invariants conserved by the equations of the active and passive scalar are the same. The importance of this conjecture warrants considerable further work to uncover its provisos and delineate its generality. It is expected that when the dynamical invariants differ, the statistical objects will differ as well.

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