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Entropy fluctuations in shell models of turbulence

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Abstract. – This letter concerns the extension to non-reversible systems of a relation between the statistics of the fluctuations of the entropy production rate and its stationary average. In the framework of a standard shell model of turbulence, we consider the statistics of the energy input rate and entropy supplied separately and provide numerical evidence that even though the fluctuations of the former quantity are consistent with the linear part of the fluctuation theorem, this is not so of the latter when the fluctuations are large enough. Moreover, in the linear regime, the slope is not in agreement with the universal value predicted by the theorem. Our results extend those of Aumaître *et al.* (*Eur. Phys. J. B*, **19** (2001) 449).

The field of non-equilibrium statistical physics is lacking the backbone principles of its equilibrium counterpart. In the recent years, some new concepts of fundamental nature have emerged. Considering the properties of the natural invariant measures of hyperbolic systems, the so-called Sinai-Ruelle-Bowen measures [1], Gallavotti and Cohen [2,3] stated their *Chaotic Hypothesis*, namely:

> A many particle [or turbulent] system in a stationary state can be regarded as a transitive Anosov system for the purpose of computing the macroscopic properties of the system.

The motivation for introducing this hypothesis is the so-called fluctuation theorem first observed numerically by Evans *et al.* [4] in the non-equilibrium stationary state of a fluid under an external shear. The general formulation subsequently provided by Gallavotti and Cohen [5] states that, if π_τ is the probability distribution of the phase space contraction rate σ_τ (identified with the entropy production rate), averaged over a time length τ, *i.e.* $\pi_\tau(p)$ denotes the probability that σ_τ is equal to $p\overline{\sigma}$, where $\overline{\sigma}$ is the infinite time average of the phase space contraction rate, then

$$\lim_{\tau\to\infty}\frac{1}{\tau}\ln\frac{\pi_\tau(p)}{\pi_\tau(-p)}=\overline{\sigma}p. \quad (1)$$

An important point in the demonstration of this relation is the assumption that the dynamics be time-reversible, namely the existence of an involution taking the time evolution to its inverse.

Thus the chaotic hypothesis has, as a direct consequence for time-reversible systems, the fluctuation relation eq. (1) with two important features: i) the linearity with respect to the fluctuating quantity p and ii) its asymptotic value which determines the slope.

Many tests of eq. (1) have been carried out within the framework of mechanically thermostated systems. Some references can be found in [6]. However, such systems are (at least *a priori*) unlikely candidates to model non-equilibrium systems, since they rely on a rather peculiar energy-pumping mechanism which, while mimicking the action of an external thermostat, ensures the convergence to a stationary state while maintaining time-reversibility at a microscopic level. This mechanism implies the identification of the phase space contraction and entropy production rates, which confers its physical relevance to eq. (1). Indeed in so-called Gaussian thermostated systems, the total energy is kept constant in the presence of an external field with the use of a Lagrange multiplier that acts as an effective damping term. The conservation of energy implies an identity between the power injected and the energy dissipated. Hence the phase space contraction and the work done or entropy production are one and the same quantity.

The natural question arises: is a form of the fluctuation relation eq. (1) relevant to the non-equilibrium statistics of irreversible systems such as, for instance, Navier-Stokes equations in the fully-developed turbulent regime? In analogy to the Gallavotti-Cohen fluctuation theorem, let P_τ denote the probability distribution of an observable assimilated to an entropy production rate whose non-equilibrium average we write $\overline{\Sigma}$. Does P_τ fluctuate in a way similar to π_τ in eq. (1)? In other words, do we have a fluctuation relation of the Gallavotti-Cohen type for P_τ,

$$\lim_{\tau\to\infty} \frac{1}{\tau} \ln \frac{P_\tau(p)}{P_\tau(-p)} = \overline{\Sigma} p \ ? \tag{2}$$

If so, the question remains: what is the fluctuating quantity to be measured? The usual dissipation mechanism of models of turbulent fluids have constant phase space contraction rates; the viscous mechanism is such that the energy dissipation rate is a strictly positive quantity, so that there is no equivalent of a fluctuation theorem for that quantity. Moreover, since those models are irreversible, the entropy produced is strictly greater than the entropy supplied. According to thermodynamics, the entropy supplied to a closed system (*i.e.* not exchanging particles with its surroundings) is equal to the ratio of the heat supplied to the temperature of the system, provided the variations of the latter are small. The former quantity is readily accessible since the heat supplied is measured by the power injected, which relies on the stirring mechanism. Meanwhile, we need to assume a well-defined temperature. But according to the usual phenomenology of turbulence, most of the energy of the system is concentrated on the integral scale, where the system is being forced. So we expect the variations of the total energy to be strongly correlated to the power injected. Hence whether the temperature for this system is well defined depends on how much work is provided by the external forcing over a given time interval in comparison to the energy of the system. Assuming the work done is small enough, we could choose to measure the temperature through the time-dependent total energy and thus have two fluctuating quantities whose ratio we identify with the entropy supplied to the system. If, on the other hand, the fluctuations of the work done were too large, we would then break away from thermodynamics and have little to say in terms of entropy supplied.

As far as experiment is concerned, tests of fluctuation relations similar to ones such as for the entropy supplied are not easy. Thus experiments have been carried out with mixed success. In [7], it was argued that the temperature fluctuations in turbulent Rayleigh-Benard convection satisfy eq. (2) as far as linearity is concerned. Yet it is hopeless to verify the universality of the slope.

Meanwhile, numerical models of turbulent flows are better suited to more comprehensive tests. In particular, shell models of turbulence appear to be good candidates. Shell models are simple in the sense that they mimic the turbulent energy cascade over a set of scales represented by discrete shells of given wave numbers and to which complex velocity variables are attached. The model consists of a set of ordinary differential equations for the velocities. The large-scale dynamics is usually driven by some forcing mechanism which generates an energy input. A non-linear quadratic interaction between neighboring shells then lets the energy flow downscales throughout an inertial range of scales where neither forcing nor dissipation is felt. This goes on until the flux reaches the smaller scales where a dissipation mechanism removes the energy. This mechanism of energy transfer is in accordance with the phenomenology of turbulence described by Kolmogorov, see [8].

In [9], Aumaître *et al.* studied the fluctuations of the power dissipated in a shell model known under the acronym of GOY after Gledzer, Ohkitani and Yamada (see [10] for references). They showed that the fluctuations of the power injected satisfy the linearity of a relation similar to eq. (2). However, the authors acknowledged that the linearity of the LHS of eq. (2) may in their case hold only for small perturbations, in contrast to the fluctuation theorem which is a large deviation result. Indeed the close-to-gaussianity of the distributions measured in [9] makes the linearity of the fluctuation relation a rather straightforward result. Moreover, a measurement of the fluctuations of the power injected is dimensionally inconsistent with eq. (2); the two sides of the equation would have different dimensions. To fix this, one needs to introduce an arbitrary dimensional factor, hence there can be no universal slope involved and therefore no theorem.

The goal of this letter is to address the issues already taken up in [9]. The fluctuation relation will be discussed both in terms of the power injected and entropy supplied to the system, which, as we indicated above, we candidly define as the ratio of the energy input rate to the total energy. We will show that the statistics of the power injected are in fact very different from that of the entropy supplied as the fluctuations grow larger. Considering the ratio on the LHS of eq. (2) for both cases, we will show that even though the linearity holds within good approximation for the fluctuations of the power supplied, this is not so for the fluctuations of the entropy supplied, for which we observe sublinear deviations as the fluctuations increase. In view of our definition of the entropy supplied, we must acknowledge that these deviations may not be relevant to the fluctuations of the actual entropy supplied and may instead reveal discrepancies with thermodynamics due to overly large fluctuations of the total energy. Nevertheless, considering the slope on the RHS of eq. (2) in the regime of small deviations, we will argue that it is not in agreement with the universal value predicted by the fluctuation relation, thus ruling out the plausibility of a fluctuation theorem for the systems under consideration. We will come back to this result in the conclusions.

We discuss the issue in the framework of the GOY shell model [10]. The model consists in a set of complex velocity variables denoted $\{u_n\}_n$ defined over a set of shells with wave numbers $k_n = k_0 2^n$, where $n = 0, \ldots, N-1$ ($u_n = 0$ outside this range). k_0 is a constant corresponding to the large scale cut-off. The evolution equations are specified as follows:

$$\begin{aligned}\frac{\mathrm{d}u_n}{\mathrm{d}t} &= ik_n\big[2u_{n+1}u_{n+2} - (\delta+1)u_{n-1}u_{n+1} + \\ &\qquad + 2^{-1}\delta u_{n-2}u_{n-1}\big]^* - \nu k_n^2 u_n + f_n\,, \end{aligned} \tag{3}$$

where the superscript $*$ denotes complex conjugation. Here the only relevant parameter is denoted by δ which belongs to the interval $-1 \leq \delta \leq 1$. We set it to $\delta = -1/2$, a popular choice for shell models of three-dimensional turbulence. The viscosity is here denoted by ν

and the external force by f_n, which we take to be concentrated on the fourth shell, $f_n = f\delta_{n,3}$, with the constant force $f = 5. \times 10^{-3}(1,1)$. In our simulations, we take $N = 28$, $k_0 = 2^{-3}$ and $\nu = 4 \times 10^{-9}$. The integration scheme is a simple Adams-Bashforth 1st-order backward differentiation scheme.

In connection to a remark made above, we point out that the phase space contraction rate is constant for the shell model defined by eq. (3). Indeed the viscosity is a constant parameter, so that the divergence of the flow is simply $-\nu \sum_n k_n^2$, which is not a fluctuating quantity. One could devise different kinds of damping mechanisms, *e.g.* replacing k_n^2 by k_n^{2p} for some integer p. All such systems are microscopically irreversible and have constant phase space contraction rates. In the context of fluid dynamics, the possibility of replacing the viscous mechanism by an effective time-dependent viscosity with the constraint that the total energy or a related quantity be kept constant has been proposed in [3]. It was studied in [11] in the framework of shell models. One could apply the chaotic hypothesis to such models and check eq. (1) for the fluctuations of the phase space contraction rate(1). However, it is not obvious that changing the damping mechanism will not affect the mechanism for the energy cascade. This question goes beyond the scope of this letter. For now we will limit our investigations to the model defined by eq. (3), where the viscosity is constant.

In this model as in other shell models, the inertial part of the evolution equation conserves two quadratic invariants. The one of interest here is the (time-dependent) energy, $E = \sum_n |u_n|^2$. Its time evolution is simply given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2\nu \sum_n k_n^2 |u_n|^2 + 2\Re(u_3 f^*). \tag{4}$$

The first term on the RHS of this equation represents the power dissipated. It gets its contribution from the larger shell indices outside the inertial range. The second one is the energy input rate, which is generated on the forcing scale. In the stationary state, the energy is conserved in average so that the power dissipated and injected will balance. Their statistics are nevertheless very different. In the sequel we will focus our attention on the statistics of the power injected either alone or divided by the energy, itself a fluctuating quantity. We note that local versions of eq. (2) can be considered if one replaces the statistics of the power injected by the local energy fluxes. In some parameter regimes, the statistics of the local fluxes have interesting features which will be studied elsewhere.

Let us denote by $\epsilon(t)$ the value of the energy input rate at some time t:

$$\epsilon(t) \equiv 2\Re[u_3(t) f^*]. \tag{5}$$

For the sake of studying the effect of time-averaging on the statistics of this quantity, we define

$$\epsilon_\tau(t_0) = \frac{1}{\tau} \int_{t_0}^{t_0+\tau} \epsilon(s)\mathrm{d}s. \tag{6}$$

Here τ should be compared to the characteristic time τ_c of the forcing scale, which can be taken to be $\tau_\mathrm{c} = 1/[k_3\sqrt{\langle |u_3|^2\rangle}]$. From our numerical data, we obtain $\tau_\mathrm{c} \approx 10$. We should point out that this definition is not unique, so that the value of the characteristic time is not as precisely defined as one might wish. However, it is not expected to vary more than by a factor of approximately 2 about our definition.

(1)Such models are relevant to the dynamical ensemble equivalence conjectured by Gallavotti [12, 13], and according to which the statistical properties of solutions of the Navier-Stokes equations in the infinite Reynolds number limit (*i.e.* the regime of fully developed turbulence) are the same as those of a reversible form of it.

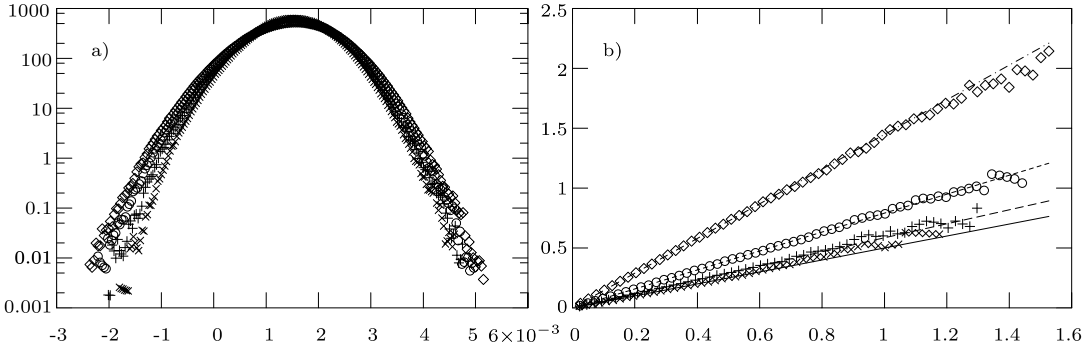

Fig. 1 – (a) Probability distributions of the time-averaged input rate ϵ_τ, where $\tau = 5$, 10, 15, 20 (τ growing inwards). (b) LHS of eq. (2) for the same data. The straight lines are linear fits obtained for the range $0 \leq p \leq 0.4$. The fits appear to match the data beyond this range, which shows that the data are consistent with a single linear regime throughout.

Figure 1 displays the probability distributions of ϵ_τ computed from a numerical simulation over times τ going from 5 to 20, that is up to approximately two large eddy turn-over times. A slight skewness is noticeable. Nevertheless, the distributions appear to be very close to Gaussian distributions when the fluctuations are not too large with respect to their standard deviation. The numerical data shown in fig. 1a) can be used in order to compute the LHS of eq. (2). This is shown in fig. 1b). As the fits show, the linearity with respect to p is quite accurately verified over the whole range of fluctuations measured. The slopes are tabulated in table I.

Consider next the probability distributions of ϵ_τ/E for the same times as above, shown in fig. 2. Notice here that the tails are very different. On a logarithmic scale, the center of the distributions are quadratic polynomials, whereas the tails are approximated by linear polynomials with slopes of opposite values on both sides. If we now use this data in order to compute the LHS of eq. (2) as shown in fig. 2b) (see table I for the slopes of the fits), we clearly see, as p increases, a significant discrepancy with the linear law observed for small fluctuations. This reflects a crossover from linear to quadratic behaviour as p increases and $p\overline{\sigma}$ moves from zero to the center of the distribution while $-p\overline{\sigma}$ goes to the left-hand tail. Based on this data, the sub-linearity could be interpreted as disproving the fluctuation theorem for the statistics of the entropy supplied. Care must be taken however, since we expect this quadratic dependence to disappear for larger deviations than those we were able to measure. Indeed for p large enough, we would be sampling the right-hand tail of the distributions which

Table I – *Averages and standard deviations of the curves displayed in figs.* 1 *and* 2*, together with the slopes of their linear fits. Notice* $p \approx 1.3$ *is on the order of three standard deviations for both data sets.*

τ	ϵ_τ			ϵ_τ/E		
	avg	std dev	slope	avg	std dev	slope
5	1.53×10^{-3}	7.78×10^{-4}	1.448 ± 0.002	2.66×10^{-2}	1.31×10^{-2}	1.376 ± 0.004
10	1.53×10^{-3}	7.47×10^{-4}	0.791 ± 0.002	2.66×10^{-2}	1.25×10^{-2}	0.734 ± 0.003
15	1.53×10^{-3}	7.08×10^{-4}	0.585 ± 0.001	2.65×10^{-2}	1.19×10^{-2}	0.538 ± 0.002
20	1.53×10^{-3}	6.66×10^{-4}	0.500 ± 0.002	2.65×10^{-2}	1.13×10^{-2}	0.451 ± 0.002

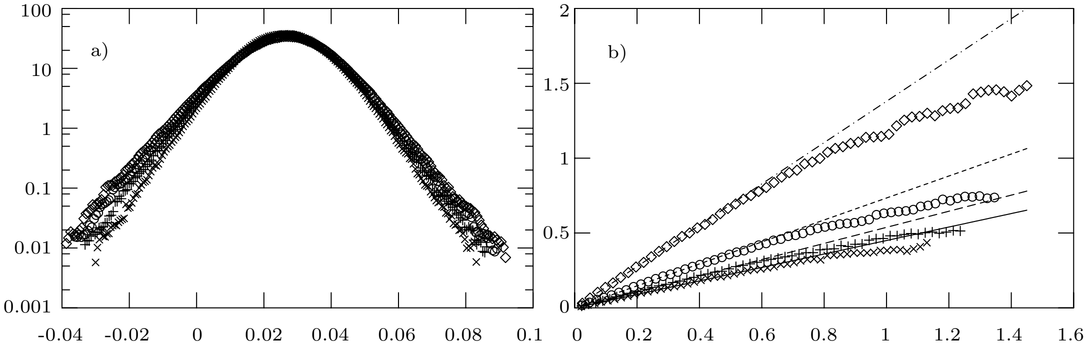

Fig. 2 – (a) Probability distributions of the time-averaged input rate ϵ_τ divided by the total energy. (b) LHS of eq. (2) for the same data. The straight lines are linear fits obtained for the range $0 \le p \le 0.4$. Here the curves appear to depart from the linear regime for larger fluctuations.

are also exponential. The linear law in eq. (2) would then depend on the difference between the decay rates of the two tails. Based on our data, an extrapolation of linear fits of the logarithms of the tails suggests the ratio in eq. (2) could saturate to a constant value for large p instead of yielding a linear law. But the lack of data at large negative p's does not allow for a more precise statement.

Whether we retrieve a linear law or not, a statement on the large fluctuations must be taken with a further grain of salt to the extent that thermodynamics may very well not be valid here in the sense that our definition of the entropy supplied in terms of the ratio of the power supplied to the energy of the system assumes that the fluctuations of the energy are small compared to the power supplied. In the absence of a well-defined temperature, there is no prescription to compute the rate of entropy supplied. Since we are measuring fluctuations averaged over extended time intervals, it is possible that the total energy varies in a way that is beyond the assumptions we made regarding the definiteness of the system's temperature.

If we now limit our analysis to small perturbations for which the fluctuations of temperature are small enough that the entropy input is well defined, we can ask whether the fluctuation relation eq. (2) holds in that range. To this end, we need to compare the slopes tabulated in the third column of table I to the average value of the entropy input rate. The numerical data yield the average value $\overline{\sigma} \simeq 2.65 \times 10^{-2}$, which is much smaller than the slopes we measured. Our numerics therefore invalidate a fluctuation relation for small fluctuations. One cannot exclude that the value given by eq. (2) will be retrieved for larger fluctuations, but this range is beyond what one can measure numerically.

To conclude, based on numerical computations, we have been able to invalidate the fluctuation relation in the framework of a shell model of turbulence. The statement concerning the universality of the slope in eq. (2) does not hold for our system, as shown by our data. Even though they did, the results on the small deviations presented in this paper are not relevant to the Gallavotti-Cohen theory. Indeed the linearity of the ratio in eq. (2) is a direct consequence of the approximation of the data by Gaussians in the appropriate limit. However, the fluctuations for which the Gallavotti-Cohen theorem holds are the large ones, not the small ones. Therefore, the slopes of the Gallavotti-Cohen relation are not related to those of the small fluctuations of the entropy supplied we measured. For large deviations we clearly see a breakdown of the linear law.

It should be noted that our results are not restricted to the GOY model. In fact very similar results can be obtained using the Sabra shell model [14]. We should also point out that the Gaussian shape of the statistics of the energy input rate is not affected if one replaces the constant forcing by a random one.

It is possible that the larger fluctuations' deviations from the linear regime are similar to those found, for instance, in the framework of injected power fluctuations in the Langevin equation [15]. However, making a similar statement for a thermodynamical quantity goes beyond our reach. We have no reliable prescription to measure an accurate rate of entropy supplied when the fluctuations of the power supplied are so large that the energy changes significantly.

We should insist that our result does not invalidate the fluctuation theorem eq. (1) itself. Indeed the system under consideration here does not fall into the class of systems for which the fluctuation relation has actually been proven. Rather we hope this work sheds some light on the limitations of its possible extensions. Whereas the gaussianity of the power injected fluctuations implies a linear law similar to that of the fluctuation theorem, the fluctuations of a properly normalized quantity such as the one we consider depart significantly from a linear law.

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