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STABLE OSCILLATIONS AND DEVIL'S STAIRCASE IN THE VAN DER POL OSCILLATOR

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A forced van der Pol relaxation oscillator is studied experimentally in the regime of stable oscillations. The variable parameter is chosen to be the driving frequency. For a range of parameter values, we show that the rotation number varies continuously from 0 to 1. This work provides experimental evidence that period-adding bifurcations to chaos previously reported by Kennedy and Chua are intimately connected to the existence of a regime of stable oscillations where the rotation number shows a Devil's-staircase structure.

1. Introduction

Period-adding bifurcations in the van der Pol oscillator were first studied experimentally and reported in 1986 by Kennedy and Chua [Kennedy & Chua, 1986]. These authors reproduced the 1927 experiment of van der Pol and van der Mark [1927]. Their motivation was to provide experimental evidence for the "period-adding route to chaos" discussed previously by Kaneko [1982, 1983]. Kaneko's study was based on a classic example of an invertible map of the circle,

$$f(x) = x + \frac{k}{2\pi}\sin(2\pi x) + b \bmod 1 \quad (|k| \leq 1)\,, \quad (1)$$

known in the literature as the (sine-) circle or standard map, which he extended to $k > 1$. In such a parameter range, the circle map is no longer invertible and shows chaotic behavior for some parameter values. As the parameter k was increased, Kaneko saw stable periodic windows of successive periods $n, n+1, \ldots$, alternating with chaotic windows. He also pointed out the existence of a period $2n+1$ window between the windows n and $n+1$, amongst others.

However, it is well known that the rotation number (the average angle by which a point is rotated per iteration of the map) of the circle map (1) is a continuous function of b for $|k| < 1$, see [Lanford, 1987]. Arjunwadkar *et al.* [1998] have shown in the context of a simple map that the continuity of the rotation number gives rise to a period-adding structure of the kind discussed by Kaneko. One might therefore argue that such period-adding bifurcations are not a route to chaos but, rather, find their origin in a nonchaotic regime of stable oscillations, and thus that Kaneko's observations were, at best, scars of the continuity of the rotation number that survive beyond $k > 1$, despite the noninvertibility of the map (1). In much the same way, we show in this paper that the period-adding bifurcations reported by Kennedy and Chua [1986] in the van der Pol oscillator originate from a regime of stable oscillations that we will identify.

The study of self-excited oscillators goes back to the late 19th century when Rayleigh [1883] considered an oscillator with negative damping,

$$m\ddot{y} + \kappa\dot{y} + \kappa'\dot{y}^3 + ky = 0, \quad (|\kappa|, |\kappa'| \ll 1)\,, \qquad (2)$$

where κ is negative and κ' positive, thereby allowing for energy to be injected into the system for small $\dot{y}$ but keeping the motion bounded due to the nonlinear term. This equation can be reduced (see e.g. [Jackson, 1991]), after derivation and taking $x = \dot{y}$, to the equation

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0, \quad (\varepsilon > 0), \tag{3}$$

considered by van der Pol [Appleton & van der Pol, 1922] and named after him. These systems are well understood [van der Pol, 1926] and have the remarkable property that the period of the oscillations changes from $T = 1$ for $\varepsilon \ll 1$ to $T \approx \varepsilon$ for $\varepsilon \gg 1$. In the latter case, the systems are referred to as nonperturbative. Their limit cycles have distorted nonelliptical shapes with regions of fast and slow motion in phase space. Energy is being stored during the slow motion and released in the fast motion phase, hence the name relaxation oscillator [Jackson, 1991].

Among the numerous systems that this equation describes, a standard one is the circuit illustrated in Fig. 1 where a neon bulb (acting as a current controlled negative resistance) is connected in parallel to a capacitor charged by a battery supply through a large resistor. Energy is stored in the capacitor in a time RC and the discharge occurs through the bulb by the ionization breakdown of the neon gas. This system thus pulses with a period $T \approx RC$.

When a periodic forcing term is added to Eq. (3), expressed as

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = E\sin(\omega t), \tag{4}$$

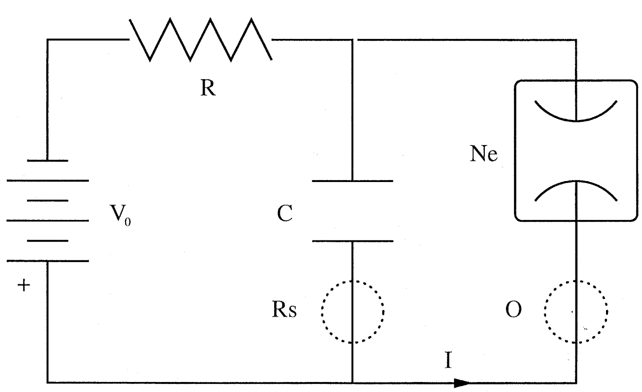


Fig. 1. An example of a relaxation oscillator. The Neon bulb (Ne) draws its energy from the capacitor (C). The charging time RC determines the period of the pulsations of the circuit. The dotted circle R_s is the location of our current monitoring resistor of 47 Ω. The dotted circle O is the location of our driving oscillator.

the behavior gets drastically complicated and a number of very interesting phenomena occur, an experimental illustration of which was given by van der Pol and van der Mark [1927]. They applied Eq. (4) to a circuit like that of Fig. 1 with a variable capacitance, and connecting an oscillator in series with the neon bulb. The frequency of the oscillator was chosen so as to match the relaxation frequency, i.e. $\omega = 1/RC$, while the amplitude was chosen to be $E = 10$ V (with the battery supply set to $V_0 = 200$ V). They reported a number of observations. First, they found that this system responds at a period which is a multiple of the driving period.[1] More specifically, as they increased the capacitance from its initial value while keeping the other parameters fixed, the period of the oscillations of the system changed discontinuously, jumping between next integer multiples of the driving period. The second observation that they did not investigate was that there are bands of apparent noise between successive periods and, third, that there is an hysteresis effect, i.e. for some values of the capacitance, there are two stable periods and the system adjusts to one or the other depending on what was the value of the capacitance preceding that transition. This paper led to a number of extensive studies of Eq. (4) and related forms, over the last seventy years or so.

The perturbative case is fairly well understood (see [Hayashi, 1964]) but the mathematics is far less tractable in the nonperturbative case. Cartwright and Littlewood [1945] and Levinson [1949] showed that Eq. (4), rewritten in the form

$$\ddot{x} - k(1-x^2)\dot{x} + x = b\lambda k \cos(\lambda t + a), \quad (k \gg 1), \tag{5}$$

admits a whole variety of solutions as the parameters are varied, some of which are stable periodic solutions, some unstable ones and some aperiodic. More specifically, for $0 < b < 2/3$, they showed that there are two sets $\{A_k\}$ and $\{B_k\}$ of alternating intervals separated by small gaps, the former where there is a pair of subharmonics of period $2\pi(2n(k)+1)/\lambda$, one stable and the other unstable, and the latter where there is, in addition, a pair of subharmonics of period $2\pi(2n(k)-1)/\lambda$. Here $n(k)$ is a large integer whose value is determined by k. Moreover, in these intervals, they also showed the existence of a family of solutions that appear to depend sensitively on the parameters. In

[1]This observation was not completely new; Rayleigh [1883] reports an experiment due to Melde involving longitudinal forced vibration of a string, whose period is the double of that of the point of attachment.

fact, as Levi [1981] pointed out, for $b \in B_k$, the invariant set is a strange attractor.

Levi [1981, 1990] showed that the geometrical properties of the Poincaré return map of Eq. (5) responsible for the indicated phenomena could essentially be understood on the basis of a continuous expanding map of the circle. In particular, for $b \in A_k$, the rotation number (here, one iteration is the time interval of the driving period) takes on a unique value whereas, for $b \in B_k$, the set of rotation numbers is exactly a closed interval.

In a numerical study of Eq. (5), Flaherty and Hoppensteadt [1978] obtained results in agreement with those of Levi [1981, 1990]. However, they also suggested that there is a regime of stable oscillations, for some range of values of k, in which the rotation number is a continuous monotone function of k whose derivative vanishes almost everywhere. Such a result is well known to be a property of one parameter family of invertible maps of the circle [Lanford, 1987], e.g. the family of maps (1) with b as the variable parameter. This therefore suggests that Levi's map of the circle [Levi, 1981, 1990] takes the yet simpler form of an invertible map.

If this is so, it should be possible to find such a region of stable oscillations experimentally. In fact, this problem is closely related to the period-adding structure Kennedy and Chua [1986] revealed in one of the noisy bands discussed by van der Pol and van der Mark in their experiment [van der Pol & van der Mark, 1927]. For stable oscillations, the rotation number takes on any rational value p/q where p and q are relative prime integers. This then implies that, as the parameter is varied, phase locking should occur for any period q (times the driving period) in an order determined by the fact that the rotation number is a monotonically increasing function of the parameter. However, it is not clear from Kennedy and Chua's data that this would indeed be the case in their experiment. Moreover, even though all the periodic windows appeared in the right order, the rotation number would only take values from 0 to 1/2 as their period-adding sequences are bounded by period-1 and period-2 windows. This therefore suggests that one should vary a parameter other than the capacitance in order to find a range of values where the rotation number varies continuously from 0 to 1.

Hayashi [1964] reported numerical results for Eq. (4), obtained by using an analog computer, where he varied both amplitude E and driving frequency ω. For E large enough, he found a sequence of subharmonics 1, 1/2, 1/3, 1/4, 1/5, 1/6. More recently Pivka *et al.* [1994] performed a numerical investigation of the driven Chua's circuit, which belongs to the class of van der Pol oscillators. They discussed in detail the period-adding law and the corresponding Devil's-staircase structure that results from varying the driving frequency. This indicates that the driving frequency might be as good a parameter as those used by Flaherty and Hoppensteadt [1978]. Moreover, it has the advantage of being very easy to implement experimentally.

In this paper, we show that the results of Flaherty and Hoppensteadt [1978] on the continuity of the rotation number are observed in the van der Pol oscillator as the experiment of Kennedy and Chua [1986] suggests. We report experimental results using the circuit of Fig. 1, with an oscillator inserted in series with the neon bulb, where, instead of varying the capacitance as in [van der Pol & van der Mark, 1927; Kennedy & Chua, 1986], we varied the driving frequency, as did Hayashi [1964] in his numerical computations. We show a range of different behaviors and, in particular, parameter regions of stable oscillations bounded by two period-1 windows, where the rotation number is found to vary continuously from 0 to 1, in the form of a Devil's staircase. We did not use the potentially more precise method of Kennedy *et al.* [1989] for determining the Devil's-staircase structure of frequency lockings. Our technique was precise enough to show clearly the connection between bifurcations and the staircase structure of stable oscillations reported here. We also point out that we did not attempt to record quasiperiodic behavior in our measurements, which the method of Kennedy *et al.* would have allowed.

2. Experimental Setup

The circuit we consider is the forced neon bulb relaxation oscillator depicted in Fig. 1. A high voltage battery V_0 (two batteries in series, type EVEREADY 467 NEDA 200 and 455 NEDA 201, nominally 67.5 V and 45 V, respectively) with large source resistance R (1 MΩ) is attached to a shunt connection of neon bulb Ne and capacitor C (2.3 μf). A sinusoidal voltage source O, (5 V peak to peak amplitude, stable to 1 mV) is attached in series with the bulb, and a small resistor R_s (50 Ω) is inserted in series with the capacitor so as to measure the current there. The driving frequency is the parameter that we chose to vary.

The oscillator we used is a HP 3325A synthesizer/function generator and the signal was analyzed with a Stanford Research Systems Model SR770 Fast Fourier Transform analyzer. The circuit turns out to be extremely sensitive to external perturbations and, in particular, to line voltage pickup. This was the reason for using batteries rather than a DC power supply. In order to further reduce the effect of such noise on the neon bulb and our measurements, the circuit was enclosed in a cast aluminium box to provide electrostatic shielding. The bulb used was a GTE type NI/2, but other bulbs gave similar behavior. In the absence of driving source, the peak currents were of order 5 mA for a pulse width of 10 μs and the average currents of order 5 μA. We also point out that the properties of the plasma inside the bulb deteriorate rather quickly for the current used, which renders repetitive measurements difficult. Within an hour of operation the natural oscillation frequency of the bulb would typically shift a few Hertz down, and after twelve hours of running the difference would be of several tens of Hertz. For this reason we did not attempt to determine the values of the frequencies at which periodic windows were observed within an accuracy better than 1/10 Hz. In the text, we will only specify these values to within 1 Hz.

3. Experimental Observations

Without the driving force, the natural oscillation frequency is observed to be $\nu_0 = 871$ Hz (to be compared with the inverse of the relaxation time $\tau = RC \simeq 2.3$ ms, $1/\tau = 435$ sec^{-1}) and therefore the system adjusts itself to the driving frequency as long as it remains close to that value (in this case, in the range 811 Hz $< \nu <$ 967 Hz). We will refer to this frequency window as the main period-1 window (meaning that the circuit responds to the driving period). We will henceforth refer to the ratio of the period of oscillations of the circuit to the driving period as the period number.

When the driving frequency goes immediately below and above these bounds, the system responds at a much longer period so that a high number of subharmonics (of the driving frequency) appear in the spectrum. Figure 2 compares the spectrum of a period-1 oscillation at the lower end of the period-1 window with that of a period-30 oscillation right below it. In practice, higher periods are difficult to lock because the widths of their windows get so small that instabilities dominate. We note that the noise level shows rather large peaks due to the line pickup signal, which appear 60 Hz above and below (indicated by arrows) the driving frequency peak of Fig. 2(a). In effect, these peaks will appear with most subharmonics and are sometimes difficult to distinguish from the actual subharmonics belonging to a specific period. We will illustrate this in Fig. 4 below. As far as our measurements suggest, in a range of frequencies above and below the main period-1 window, all the parameter values correspond to stable periodic oscillations. Instabilites due to imperfection of

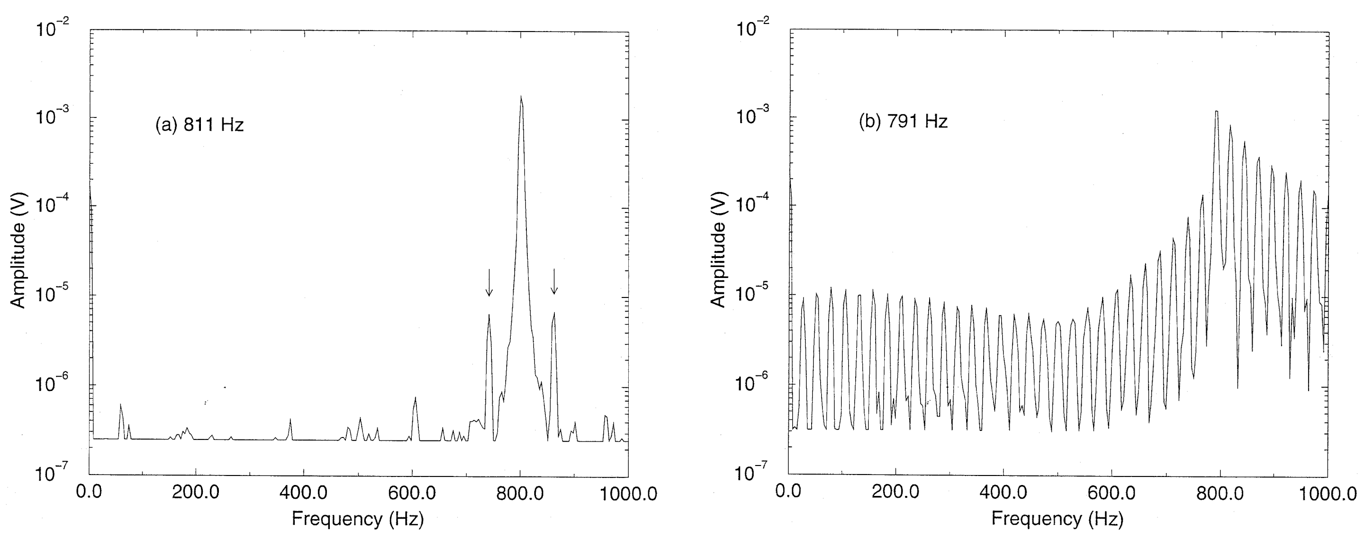
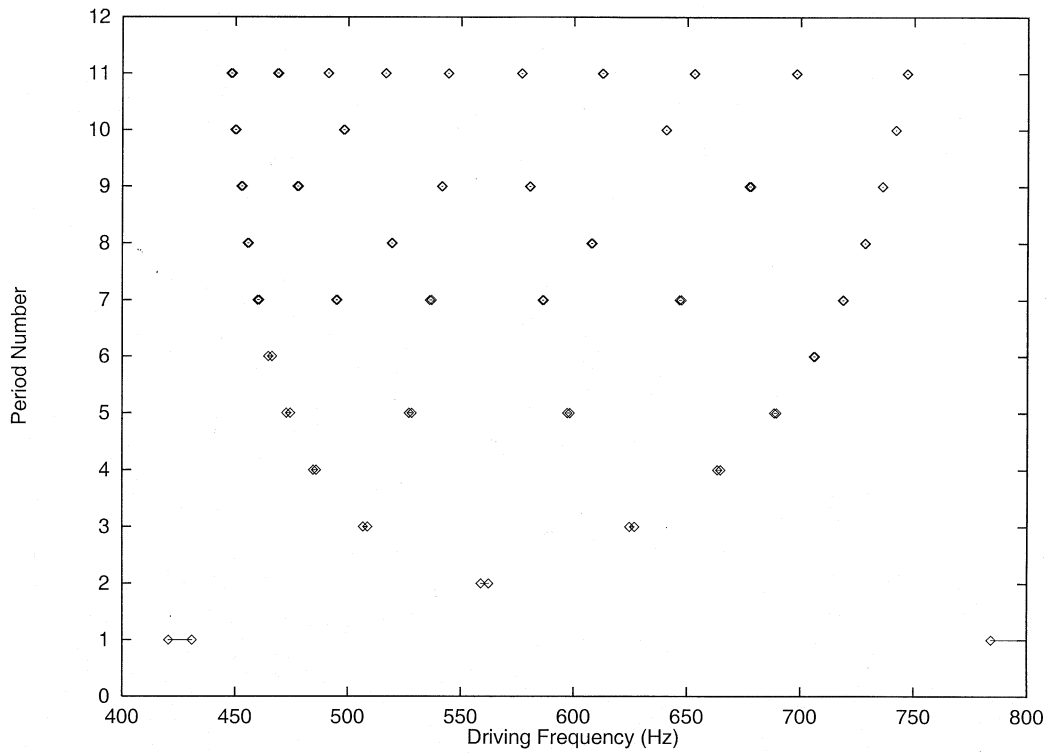
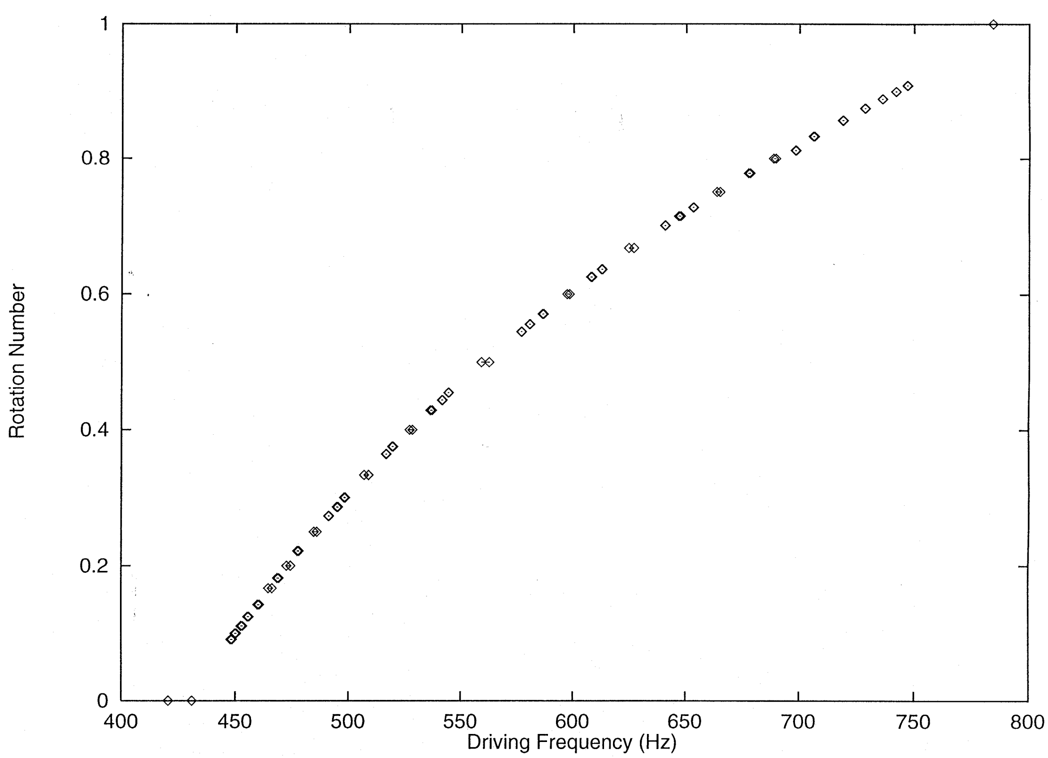


Fig. 2. (a) The lower end of the period-1 window at $\nu = 811$ Hz. The arrows are indicating the two 60 Hz peaks below and above the driving frequency peak. (b) A period-30 observed at $\nu = 791$ Hz.



(a)



(b)

Fig. 3. (a) The period-adding structure of the stable oscillations between the two main period-1 windows and (b) the corresponding Devil's staircase of the rotation numbers.

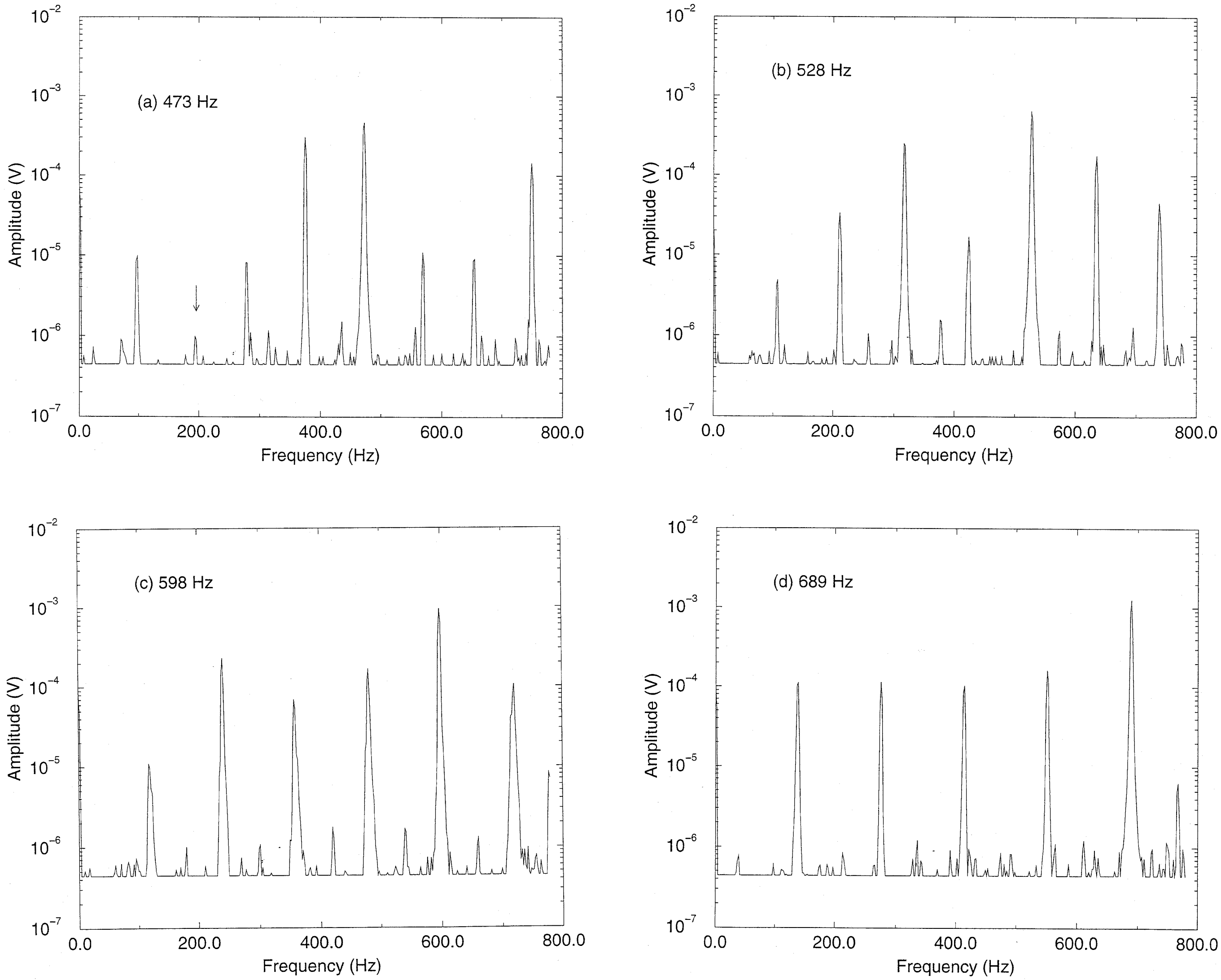


Fig. 4. The spectra of the four period-5 windows of Fig. 3. (a) $\nu = 473$ Hz (the arrow indicates the second subharmonic), (b) $\nu = 528$ Hz, (c) $\nu = 598$ Hz, (d) $\nu = 689$ Hz.

our system are clearly distinguishable from the chaotic regime in that the latter gives rise to a continuous spectrum with fluctuating amplitudes.

Below the main period-1 window, different sequences of smaller periodic windows can be observed, as in Fig. 3(a). The main sequence corresponds to the period numbers decreasing from very large period numbers by unit steps until a period-2 window is reached and then increasing, again by unit steps, to very high period numbers until the sequence stops and the period falls to 1 at 431 Hz. Note that the new period-1 window appears at a driving frequency about a half of the natural oscillation frequency. This result should be contrasted with that reported by Hayashi [1964], for perturbative van der Pol oscillators, which predicts a period-1/2 at that frequency. Besides the sequence of periodic windows just described, a number of other periodic windows appear between successive windows of the sequence. For instance, period-9 appears between periods-5 and -4. In Fig. 3(a) we have recorded periodic windows up to period-11. We point out that the number of occurences of each period and their order is consistent with a rotation number varying continuously from 0 to 1. Actually, these period numbers are most easily converted to rotation numbers by assuming monotonicity of the rotation number as a function of the parameter. For instance, the four period-5 windows at 473 Hz, 528 Hz, 598 Hz and 689 Hz,

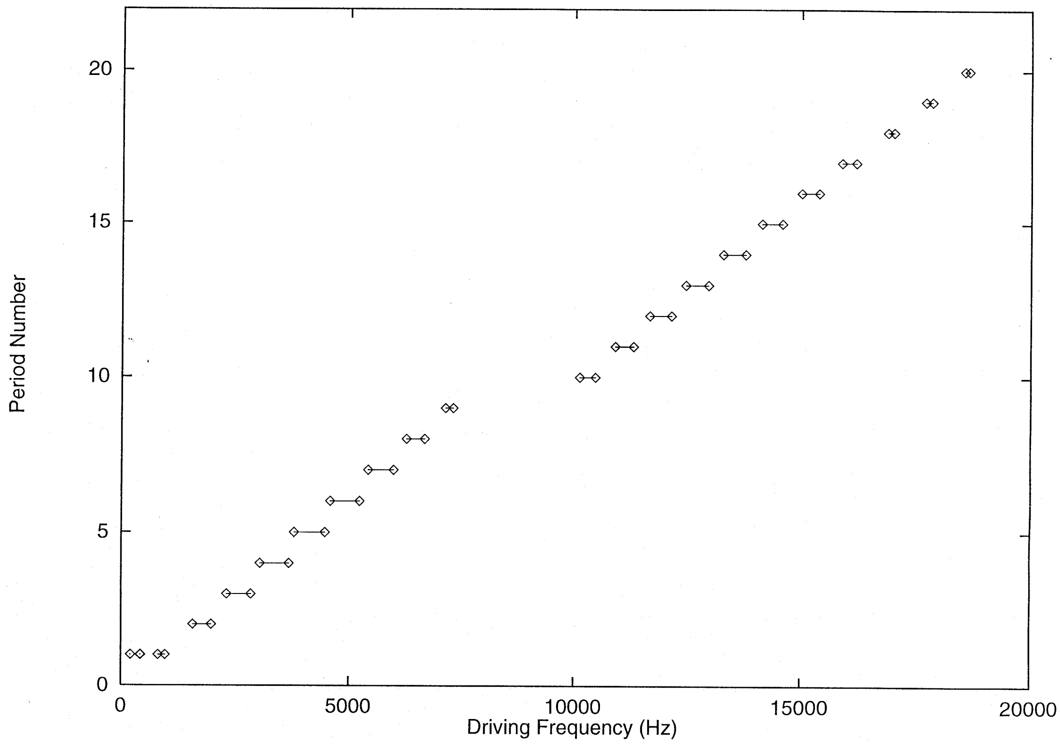


Fig. 5. The main period-adding sequence. Notice the discontinuity at the transition to period-10.

whose respective spectra are shown in Fig. 4, correspond, from left to right, to rotation numbers 1/5, 2/5, 3/5, 4/5. We thus obtain the Devil's staircase of Fig. 3(b). Figure 4(a) is a good example of the problems that 60 Hz peaks cause in the measurement. Indeed the second subharmonic (indicated by an arrow) has an amplitude that is about two decades less than the driving frequency peak and which makes it of the same magnitude as the 60 Hz peaks below and above the fourth subharmonic.

If the driving frequency is decreased further below the period-1 window at 431 Hz, a new sequence of period-adding bifurcations is observed, until another period-1 window is reached at 212 Hz, which is about a fourth of the natural oscillation frequency. We point out however that only up to two subharmonics distinctively appear in the spectrum and it is therefore difficult to read off the successive periods. Below 212 Hz, no subharmonics are observed but the existence of higher periods can be inferred from higher harmonics that are not multiples of the driving frequency. Again a period-1 window distinctively appears at about an eighth of the natural oscillation frequency.

If the driving frequency is increased above the main period-1 window, a type of structure of periodic windows similar to Fig. 3 is observed, which however stops after the period-2 window is reached, and thus corresponds to a rotation number increasing monotonically from 0 to 1/2. This region shares similarities with the period-adding sequence reported in [Kennedy & Chua, 1986].

The period-2 window occurs at a value of the driving frequency approximately equal to twice the natural oscillation frequency ν_0, so that there is a subharmonic that lies in its vicinity. In the same way, a period-3 window is found at a driving frequency approximately equal to three times the frequency ν_0 and so on. The resulting pattern is shown in Fig. 5. We will refer to this as the main period-adding sequence. Except for a discontinuous jump between the periods-9 and -10, it is roughly linear with respect to the driving frequency. We note that this figure is much like the ones given in [van der Pol & van der Mark, 1927; Kennedy & Chua, 1986]. There are some differences though; for instance, successive stable periodic windows are separated by some kind of transition window (which we will discuss below) and we do not find hysteresis effects of the kind reported in these two references.

Chaotic features similar to those reported by Kennedy and Chua [1986] start showing up beyond

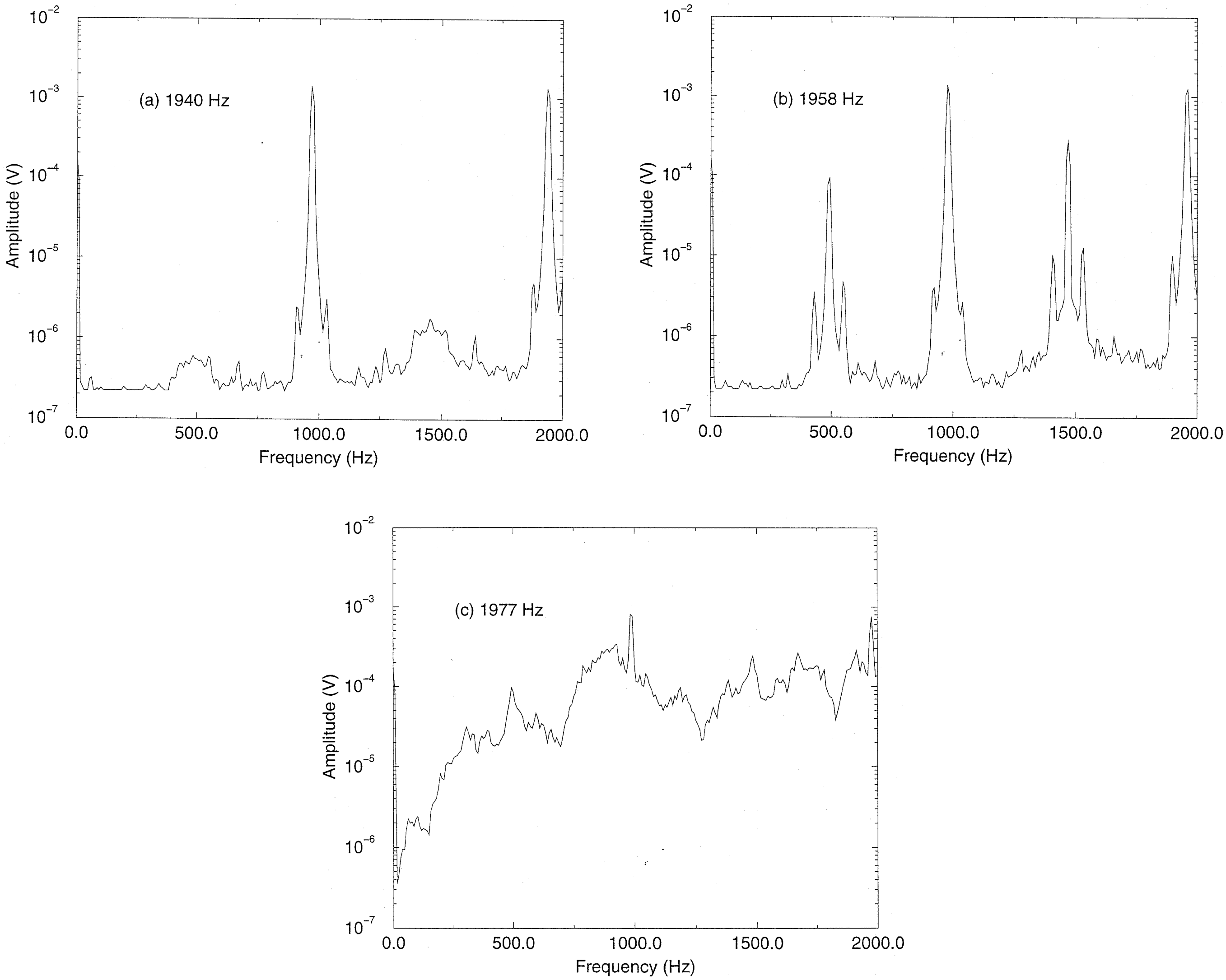


Fig. 6. The spectra of the period-doubling sequence $2 \to 4 \to$ chaos. (a) Period-2 at $\nu = 1940$ Hz, (b) period-4 at $\nu = 1958$ Hz, (c) chaos at $\nu = 1977$ Hz.

the main period-2 window. Thus, in the transition between the period-2 and period-3 windows, we still have period-adding but, this time, mixed with period-doubling transitions to chaotic oscillations; the transitions observed are the following (see Fig. 6): $2 \to 4 \to$ chaos $\to 5 \to 10 \to$ chaos $\to 8 \to 16 \to$ chaos $\to 11 \to 22 \to$ chaos $\to 3$. The next few transitions, period-3–period-4 windows up to period-9–period-10 windows, show only period-doubling transitions to chaos: $n \to 2n \to$ chaos $\to n+1$. Further up in the frequency range, we note that for the periods 10 and above the windows are shifted: The first subharmonic is not in the range of ν_0, but rather above 200 Hz. The corresponding windows sketched in Fig. 5 have some structure to them, typically a period-n window contains transitions to $2n$, $4n$ and then decreases back to $2n$ and n, which remains until a sharp transition to a chaotic regime occurs. The frequency at which this transition occurs varies depending on whether we are increasing or decreasing the driving frequency, i.e. it shows hysteresis.

4. Discussion and Conclusions

We have investigated a simple version of a forced van der Pol's oscillator and showed that, in a range of the driving frequency between one and a half times the natural oscillation frequency, a regime of stable oscillations is observed, where the rotation number varies from 0 to 1. For frequencies above twice the natural oscillation frequency, chaotic oscillations are observed and, correspondingly, the rotation number varies discontinuously from one stable periodic orbit to the next.

The Devil's staircase of Fig. 3 should be compared to the staircase discussed by Pivka *et al.* [1994] in the Chua's circuit. The similarities between the successive periodic windows were found as well as the absence of chaos points towards the universality of stable oscillation regimes in van der Pol oscillators. We therefore hope that this work sheds new light on the simplicity that underlies the dynamics of nonperturbative van der Pol oscillators. As we mentioned earlier, a regime of stable oscillations such as the one we found for the lower range of the driving frequency indicates that the dynamics on the attractor should be understandable on the basis of a smooth invertible map of the circle.

Parameter regimes where chaos and period-adding coexist are far less understood. Kaneko [1982, 1983] suggests that phase locking occurs at rational rotation numbers in a way similar to the invertible case. However this is not generally true. For instance, when period-doubling sequences appear, the rotation number jumps from $1/n$ to $1/2n$. Moreover, it is well known that the phase-locked motions (Arnol'd tongues) overlap for $k > 1$ in Eq. (1), which implies that several different periodic oscillations can occur for given (k, b), depending on the initial conditions [Baker & Gollub, 1996]. However, what can happen is that some sequences are preserved as, for instance, the sequence $3n - 1$ we found between the periods-2 and -3 of the main sequence, and where each n-period doubles.

Period-adding bifurcations in chaotic systems have been investigated in [Nusse & Yorke, 1992; Nusse *et al.*, 1994; Chin *et al.*, 1994], who classify them under border collision bifurcations. However, this approach does not address the connections between these bifurcations and the properties of the rotation number. As our experiment suggests, there will be much to learn from an approach combining these different views.

Acknowledgments

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