Algebraic two-level convergence theory for singular systems

Yvan Notay*

Service de Métrologie Nucléaire
Université Libre de Bruxelles (C.P. 165/84)
50, Av. F.D. Roosevelt, B-1050 Brussels, Belgium.
email: ynotay@ulb.ac.be

Report GANMN 15-01

July 2015

Abstract

We consider the algebraic convergence theory that gives its theoretical foundations to classical algebraic multigrid methods. All the main results constitutive of the approach are properly extended to singular compatible systems, including the recent sharp convergence estimates for both symmetric and nonsymmetric systems. On the other hand, issues associated with singular coarse grid matrices are discussed in detail.

Key words. singular system, multigrid, multilevel, convergence analysis, preconditioning, AMG

AMS subject classification. 65F08, 65F10, 65F50

*Yvan Notay is Research Director of the Fonds de la Recherche Scientifique – FNRS
1 Introduction

In this paper, we consider two level iterative solution methods for compatible singular linear systems

\[ Au = b; \quad b \in R(A). \]  

(1.1)

Such systems arise in numerous applications; they may be associated with Markov chains, discretized partial differential equations with Neumann boundary conditions, Laplacian of graphs, etc. In many cases, the systems are large and badly conditioned. In such a context, multilevel algorithms are often methods of choice [31, 37, 38], and these have effectively been successfully applied to singular systems in a number of works; see, e.g., [3, 10, 11, 12, 21, 35, 36, 39].

For general iterative methods, theoretical issues associated with singularities have been studied for a long time and are now fairly well understood. However, regarding multilevel methods, most previous works focus on practical aspects, and the state of the art is less advanced with respect to theory. A noteworthy contribution is the one in [3], where McCormick’s bound [23] is extended to symmetric semidefinite matrices. However, whereas this theory allows to analyze multilevel algorithms of V-cycle type, it does not yield sharp two-level convergence estimates [25]. The analysis in [20] is also dedicated to multi-level convergence estimates which are by nature less tractable than bounds for the two-level case.

Two-level analysis is actually more widely used even for methods that involve many levels in practice. This is particularly true for the algebraic analysis initiated by Brandt [4] and which inspired many works on algebraic multigrid (AMG) methods; see, [6, 9, 26, 28, 30] for examples. Moreover, the original theory has been improved in, e.g., [6, 30, 32], yielding finally sharp bounds in [13, 14] for symmetric positive definite matrices, whereas the approach has been extended to nonsymmetric matrices in [27] (We refer [22, 29] for extensive reviews). Nevertheless, the extension of these results to singular systems has not been discussed so far. This is the main motivation of this paper, where we address the involved issues and provide a generalization of all the main results constitutive of the approach.

We also analyze in detail a peculiarity of multilevel algorithms for singular systems: often, the coarse grid matrix is “naturally” singular as well, raising several questions. The first one is the compatibility of the coarse systems that are to be solved during the course of the iterations, which is effectively addressed in most works on the topic. However, once this is guaranteed, a specific generalized inverse is mostly chosen without much discussion or comment; this is often the Moore-Penrose inverse (e.g., [3, 36]), but not always [39]. Here we take a wider viewpoint and analyze whether the choice of the generalized inverse has or not an influence on the following of the iterations.

The remainder of this paper is organized as follows. In Section 2, we recall some basic facts about the iterative solution of singular systems. In Section 3, we develop our analysis; more precisely, we first state the general framework (Section 3.1), and next we analyze the influence of the generalized inverse that defines coarse grid correction (Section 3.2); we
then proceed (Section 3.3) with our extension to the singular case of the main results in [27] that apply to general (nonsymmetric) matrices; thereafter, we discuss the particular cases of matrices positive semidefinite in \( \mathbb{R}^n \) (Section 3.4) and symmetric positive semidefinite (Section 3.5). Finally, conclusions are drawn in Section 4.

**Notation**

For any matrix \( C \), \( \mathcal{R}(C) \) denotes its range and \( \mathcal{N}(C) \) its null space. If \( C \) is square, \( \sigma(C) \) denotes its spectrum. If \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are two subspaces, \( \mathcal{W}_1 + \mathcal{W}_2 \) denotes their (not necessarily direct) sum; i.e., the set of linear combinations of vectors from \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \).

**2 General iterative methods for singular systems**

As mentioned in the preceding section, there is a rich literature on the iterative solution of singular systems. Here we only recall the results that are useful for our later developments, or that help to put them in proper perspective. We also analyze in detail the conditions under which two different preconditioners may be considered equivalent, because this is crucial for a correct understanding of our results in the next section.

Noting \( \mathcal{B} \) the preconditioning matrix, stationary iterations to solve (1.1) correspond to

\[
\mathbf{u}_{m+1} = \mathbf{u}_m + \mathcal{B}(\mathbf{b} - A \mathbf{u}_m), \quad m = 0, 1, \ldots.
\]

(2.1)

Often instead of \( \mathcal{B} \) one writes \( \mathcal{B}^{-1} \), where \( \mathcal{B} \) is the preconditioner or the main operator in a splitting of \( A \). However, in the singular case, \( \mathcal{B} \) does not need to be regular, hence using \( \mathcal{B}^{-1} \) would entail some loss of generality.

Letting \( \mathbf{u} \) be any particular solution to (1.1), there holds

\[
\mathbf{u} - \mathbf{u}_{m+1} = (I - \mathcal{B} A) (\mathbf{u} - \mathbf{u}_m);
\]

that is,

\[
\mathbf{u} - \mathbf{u}_m = T^m (\mathbf{u} - \mathbf{u}_0),
\]

where

\[
T = I - \mathcal{B} A
\]

(2.2)

is the iteration matrix.

It is well known (see [18] for an early reference) that \( \mathbf{u}_m \) converges to a particular solution for any \( \mathbf{b} \in \mathcal{R}(A) \) if and only if \( T^m \) converges to a projector onto \( \mathcal{N}(A) \). In, e.g., [8, Theorem 2.1], this is shown to hold if and only if the three following conditions are satisfied: \( \mathcal{N}(\mathcal{B} A) = \mathcal{N}(A) \), 1 is a semi-simple eigenvalue of \( T \), and all other eigenvalues are less than 1 in modulus. This yields the following lemma, observing that the first two conditions are equivalent to the assumption that the eigenvalue 0 of \( \mathcal{B} A \) has algebraic multiplicity equal to \( k \), where \( k = \dim(\mathcal{N}(A)) \). (See [33] for a further discussion of this topic).
Lemma 2.1. Let $A$ and $B$ be $n \times n$ matrices, and let $k = \dim(\mathcal{N}(A))$. The iterative scheme defined by (1.1) converges to a particular solution of (1.1) for any $b \in \mathcal{R}(A)$ if and only if the following two conditions hold:

1. The algebraic multiplicity of the eigenvalue 0 of $BA$ is equal to $k$;
2. $\rho_{\text{eff}} = \max_{\lambda \in \mathcal{R}(BA)} |1 - \lambda| < 1$.

Krylov subspace methods have also been widely studied in presence of singularity; see, e.g., [7, 15, 17]. Here we only recall the main conclusions from [15] about GMRES (the conjugate gradient method is discussed below together with the positive semidefinite case). The following lemma is based on the observation that the condition (1) in Lemma 2.1 is in fact sufficient to apply the framework developed in Section 2.5 of [15], where a GMRES process for a singular system is decomposed into the components belonging to the range of the matrix and the components orthogonal to it.

Lemma 2.2. Let $A$ and $B$ be $n \times n$ matrices, and let $k = \dim(\mathcal{N}(A))$. If the algebraic multiplicity of the eigenvalue 0 of $BA$ is equal to $k$, then GMRES iterations to solve (1.1) are well defined with either left or right preconditioning $B$. For any $b \in \mathcal{R}(A)$, the convergence of such a GMRES process is the same as that of

(a) in case of left preconditioning, a regular GMRES process with system matrix $Q_B^TBAQ_B$, where $Q_B$ is a matrix whose columns form an orthonormal basis of $\mathcal{R}(BA)$;

(b) in case of right preconditioning, a regular GMRES process with system matrix $Q_A^TABAQ_A$, where $Q_A$ is a matrix whose columns form an orthonormal basis of $\mathcal{R}(AB)$.

Moreover, the eigenvalues of both matrices $Q_B^TBAQ_B$ and $Q_A^TABAQ_A$ are, counting multiplicities, equal to the $n - k$ nonzero eigenvalues of $BA$.

Proof. In Section 2.5 of [15] it is shown how GMRES for a compatible system can be decomposed into the components belonging to the range of the (preconditioned) matrix, and those orthogonal to it. This decomposition provides the required results as soon as it is applicable, that is, as soon as $Q_B^TBAQ_B$ (case (a)) or $Q_A^TABAQ_A$ (case (b)) are nonsingular. But, by [16, Theorem 1.3.22], $Q_B^TBAQ_B$ has the same nonzero eigenvalues (counting multiplicities) as $Q_B^TQ_B^TBA$, which is itself equal to $BA$ since $Q_B^TQ_B$ is a projector onto $\mathcal{R}(BA)$. Since our assumptions imply that $B$ has exactly $n - k$ nonzero eigenvalues, where $n - k$ is also the row and column size of $Q_B^TBAQ_B$, the conclusions follows for (a). Similarly, one finds that $Q_A^TABAQ_A$ has the same nonzero eigenvalues as $ABA$, which by virtue of [16, Theorem 1.3.22] again, also coincide with the nonzero eigenvalues of $BA$.

On the other hand, given an iteration matrix $T$ (or, equivalently, a preconditioned matrix), one may wonder about the existence and the uniqueness of a regular preconditioner $B$ such that $T = I - B^{-1}A$. This question is analyzed in Theorems 2.1 and 2.2 of [2], and we reproduce them partly below for further reference.

Lemma 2.3. Let $A$ and $T$ be $n \times n$ matrices. If $A$ is singular and $\mathcal{N}(I - T) = \mathcal{N}(A)$, then there exist infinitely many nonsingular matrices $B$ such that $T = I - B^{-1}A$. 

4
Note that if $T$ is a convergent iteration matrix (in the sense of Lemma 2.1) whereas \( \mathcal{N}(A) \subset \mathcal{N}(I-T) \), then the condition \( \mathcal{N}(I-T) = \mathcal{N}(A) \) holds, since, by (1) of Lemma 2.1, the geometric multiplicity of the eigenvalue 0 of \( I-T \) cannot exceed \( \dim(\mathcal{N}(A)) \).

This result recalls us that different preconditioners may be in fact equivalent in the singular case. The equivalence is, however, not restricted to cases where the corresponding preconditioned matrices are identical. Indeed, considering either stationary iterations or Krylov subspace methods for compatible systems, it is clear that the preconditioning is applied only to vector \( r \) in \( \mathcal{R}(A) \). Thus, only the action on vectors of that subspace matters. Moreover, if two preconditioners applied to a same vector provide the same result up to a component in \( \mathcal{N}(A) \), this latter influences which particular solution will be finally obtained, but does otherwise not influence the following of the iterations. Indeed, subsequent multiplication by \( A \) will ignore this component; hence, e.g., GMRES parameters are not affected and the next residual vector will be the same. In the following lemma, we give conditions under which such extended equivalence holds, and we furthermore show that, under these conditions, the preconditioned matrices have same spectrum.

**Lemma 2.4.** Let \( A, B_1, \) and \( B_2 \) be \( n \times n \) matrices. If
\[
\pi_{\mathcal{R}(A^T)}^T B_1 \pi_{\mathcal{R}(A)} = \pi_{\mathcal{R}(A^T)}^T B_2 \pi_{\mathcal{R}(A)} ,
\]
where \( \pi_{\mathcal{R}(A^T)} \) and \( \pi_{\mathcal{R}(A)} \) are projectors onto, respectively, \( \mathcal{R}(A^T) \) and \( \mathcal{R}(A) \), then, for any \( r \) in \( \mathcal{R}(A) \),
\[
B_1 r - B_2 r \in \mathcal{N}(A) .
\]
Moreover, \( B_1 A \) and \( B_2 A \) have the same eigenvalues (counting multiplicity).

**Proof.** One has, since \( r \in \mathcal{R}(A) \) implies \( \pi_{\mathcal{R}(A)} r = r \),
\[
B_1 r - B_2 r = (B_1 - B_2) \pi_{\mathcal{R}(A)} r
= \pi_{\mathcal{R}(A^T)}^T (B_1 - B_2) \pi_{\mathcal{R}(A)} r + (I - \pi_{\mathcal{R}(A^T)}^T) (B_1 - B_2) \pi_{\mathcal{R}(A)} r ,
\]
and the last right hand side belongs to \( \mathcal{N}(A) \) because \( \pi_{\mathcal{R}(A^T)}^T (B_1 - B_2) \pi_{\mathcal{R}(A)} = 0 \) by assumption, whereas, \( \mathcal{R}(I - \pi_{\mathcal{R}(A^T)}) = \mathcal{N}(\pi_{\mathcal{R}(A^T)}^T) = \mathcal{R}(\pi_{\mathcal{R}(A^T)})^\perp = \mathcal{R}(A^T)^\perp = \mathcal{N}(A) \). The statement about the eigenvalues follows for the fact that, for \( i = 1, 2 \), there holds that \( B_i A = B_i \pi_{\mathcal{R}(A)} A \pi_{\mathcal{R}(A)}^T \) has, by [16] Theorem 1.3.22], the same eigenvalues as \( \pi_{\mathcal{R}(A^T)}^T B_i \pi_{\mathcal{R}(A)} A \).

**The semidefinite case**

An important particular case is when \( A \) is symmetric and positive semidefinite. Then, the convergence can be analyzed with respect to the seminorm \( \| \cdot \|_A = (\cdot, A \cdot)^{1/2} \), and it is easy to see that \( \| u - u_m \|_A \) does not depend on the particular solution \( u \). Moreover, if the preconditioning matrix \( B \) is symmetric and positive definite (SPD), condition (1)
of Lemma 2.1 is automatically satisfied, whereas one may consistently use the preconditioned conjugate gradient method [17]. These observations are summarized in the following lemma, where we also use the fact that, by Lemma 2.4, the requirement on the preconditioning $B$ can be relaxed: $B$ needs only to be equivalent to some SPD preconditioning matrix.

**Lemma 2.5.** Let $A$ be a symmetric positive semidefinite $n \times n$ matrix, and let $B$ be a $n \times n$ matrix such that

$$\pi_{\mathcal{R}(A)} B \pi_{\mathcal{R}(A)} = \pi_{\mathcal{R}(A)} \hat{B} \pi_{\mathcal{R}(A)} , \quad (2.3)$$

for some SPD preconditioning matrix $\hat{B}$, where $\pi_{\mathcal{R}(A)}$ is the orthogonal projector onto $\mathcal{R}(A)$. Then $BA$ has, counting multiplicities, $k$ times the eigenvalue 0 and $n-k$ positive eigenvalues, where $k = \text{dim}(\mathcal{N}(A))$.

Moreover, considering the linear system (1.1) and letting $u$ be any particular solution, the following propositions hold.

1. When performing stationary iterations (2.1),

$$\|u - u_m\|_A \leq (\rho_{\text{eff}})^m \|u - u_0\|_A \quad (2.4)$$

where

$$\rho_{\text{eff}} = \max_{\lambda \in \sigma(BA), \lambda \neq 0} |1 - \lambda|. \quad (2.5)$$

2. When performing conjugate gradient iterations with preconditioning $B$, the convergence is the same as that of a regular conjugate gradient process in $\mathbb{R}^{n-k}$ with SPD preconditioning and for which the eigenvalues of the preconditioned matrix are equal to the nonzero eigenvalues of $BA$. Moreover,

$$\|u - u_k\|_A \leq 2 \left( \frac{\sqrt{\kappa_{\text{eff}}} - 1}{\sqrt{\kappa_{\text{eff}}} + 1} \right)^k \|u - u_0\|_A \quad (2.6)$$

where

$$\kappa_{\text{eff}} = \frac{\max_{\lambda \in \sigma(BA), \lambda \neq 0} \lambda}{\min_{\lambda \in \sigma(BA), \lambda \neq 0} \lambda}. \quad (2.7)$$

3 Analysis of two-level preconditioning

3.1 General framework

We assume that the system matrix $A$ is a real $n \times n$ matrix, and we consider the two-level schemes that are described by the iteration matrix

$$T_{TL} = (I - M_2^{-1} A)^{\nu_2} (I - P A^g R A) (I - M_1^{-1} A)^{\nu_1}, \quad (3.1)$$
where $M_1$ (resp. $M_2$) is a nonsingular $n \times n$ matrix which defines the pre-smoother (resp. post-smoother), and where $\nu_1$ (resp. $\nu_2$) is the corresponding number of smoothing steps. $P$ is the prolongation matrix, of size $n \times n_c$ with $n_c < n$, $R$ is the restriction matrix, of size $n_c \times n$, and $A_c$ is the coarse grid matrix; we restrict ourselves to Galerkin coarse grid matrices; i.e., we assume

$$A_c = R A P ,$$

but, we do not assume that $A_c$ is invertible, and in (3.1) the notation $A_c^g$ stands for a generalized inverse of $A_c$; that is, see, e.g., [1], any matrix satisfying

$$A_c A_c^g A_c = A_c .$$

This definition is motivated as follows: since (3.3) implies

$$A_c A_c^g r_c = r_c \quad \forall r_c \in \mathcal{R}(A_c) ,$$

the vector $A_c^g r_c$ is indeed a particular solution to any compatible system $A_c u_c = r_c$. Clearly, if $A_c$ is singular, there are infinitely many generalized inverses, and the iterative process may also be influenced by the choice of the latter. This will be addressed in the next subsection, where we also discuss conditions under which the coarse grid systems are always compatible (i.e., conditions under which, during the course of the iterations, $A_c^g$ is effectively applied only to vectors in $\mathcal{R}(A_c)$).

Now, to develop our analysis, we need an explicit expression of the preconditioning matrix corresponding to (3.1); i.e., of the matrix $B_{TL}$ such that $I - B_{TL} A = T_{TL}$. As seen in the preceding section, this latter equation does not uniquely define $B_{TL}$ in the singular case, but which one we pick up does not matter since all our results refer to $B_{TL} A$ and not to $B_{TL}$ alone. Here we use

$$B_{TL} = Y^{-1} + (I - M_2^{-1} A)^{\nu_2} P A_c^g R (I - A M_1^{-1})^{\nu_1} ,$$

and one may check that $I - B_{TL} A = T_{TL}$ indeed holds if $Y$ is a nonsingular matrix satisfying

$$I - Y^{-1} A = (I - M_2^{-1} A)^{\nu_2} (I - M_1^{-1} A)^{\nu_1} .$$

Here again, $Y$ is not unique, but it is interesting to observe that, as seen above (Lemma 2.3 and the subsequent discussion), the existence of such $Y$ is guaranteed as soon the right hand side of (3.5) defines a convergent iterative process. Finally, it is worth noting that in this case, the matrix

$$(I - M_1^{-1} A)^{\nu_1} (I - M_2^{-1} A)^{\nu_2}$$

also defines a convergent iteration, which guarantees the existence of a nonsingular matrix $X$ satisfying

$$I - X^{-1} A = (I - M_1^{-1} A)^{\nu_1} (I - M_2^{-1} A)^{\nu_2} .$$

$^{1}\nu_1$, $\nu_2$ are nonnegative integers, $\nu_1 = 0$ (resp. $\nu_2 = 0$) corresponding to no pre-smoothing (resp. no post-smoothing)
3.2 The coarse grid correction

The coarse grid correction step corresponds to the term \( PA^g R \) in (3.1) or (3.4): given the residual vector \( \tilde{r} \) obtained after pre-smoothing, in consists in the following substeps.

1. Restrict \( \tilde{r} \) on the coarse grid: 
   \[ r_c = R \tilde{r} \]
2. Solve the coarse problem \( A u_c = r_c \):
   \[ u_c = A^g r_c \]
3. Prolongate \( u_c \) on the fine grid: 
   \[ \tilde{u} = P u_c \]

In the regular case, \( A_c \) is logically assumed regular as well and hence \( A^g_c = A^{-1}_c \). In the singular case, however, \( A_c \) is often “naturally” singular. Indeed, as is well known, multigrid methods are most efficient when near kernel modes are included in the range of the prolongation. Then, applying this principle to a singular problem, the range of the prolongation will contain the null space of the matrix, entailing the singularity of \( A_c = R A P \) since \( A P v_c = 0 \) for all \( v_c \) such that \( P v_c \in \mathcal{N}(A) \). Similarly, \( R^T v_c \) will be in the left null space of \( A_c \) for any \( v_c \) such that \( R^T v_c \in \mathcal{N}(A^T) \).

Fortunately, such singular modes are harmless. Indeed, on the one hand, if some \( r \) is in the range of \( A \), there holds \( n^T r = 0 \) for all \( n_L \in \mathcal{N}(A^T) \), and hence \( v^T(R r) = 0 \) for any \( v_c \) such that \( R^T v_c \in \mathcal{N}(A^T) \). Because, during the course of iterations, only residual vectors belonging to \( \mathcal{R}(A) \) are restricted on the coarse grid, it then follows that coarse grid systems to be solved are always compatible when all left kernel vectors of \( A_c \) are inherited from left kernel vectors of \( A \) in the range of \( R^T \).

On the other hand, if, similarly, all right kernel vectors of \( A_c \) are inherited from right kernel vectors of \( A \) in the range of \( P \), all null space components on the coarse grid (i.e., all vectors in \( \mathcal{N}(A_c) \)) are prolonged into null space components on the fine grid, which, as seen in the preceding section, do not influence the convergence process. Therefore, which solution of the coarse grid system is picked up by the specific choice of \( A^g_c \) does not really matter.

These observations amount to state that the coarse grid correction term \( PA^g R \) is independent of the choice of \( A^g_c \) when multiplied to the left by a projector onto \( \mathcal{R}(A^T) \) and to the right by a projector onto \( \mathcal{R}(A) \). They are formally proved in the following Lemma, where we use the condition

\[
\dim(\mathcal{R}(P) \cap \mathcal{N}(A)) = \dim(\mathcal{R}(R^T) \cap \mathcal{N}(A^T)) = \dim(\mathcal{N}(A_c)) \quad (3.7)
\]

as mathematical formulation of the assumption that all vectors in \( \mathcal{N}(A_c) \) correspond to null space vectors of \( A \) in the range of \( P \), and all vectors in \( \mathcal{N}(A^T_c) \) correspond to left null space vectors of \( A \) in the range of \( R^T \). Note that \( \dim(\mathcal{N}(A_c)) \) is in fact at least as large as both \( \dim(\mathcal{R}(P) \cap \mathcal{N}(A)) \) and \( \dim(\mathcal{R}(R^T) \cap \mathcal{N}(A^T)) \). Thus, (3.7) amounts to assume that \( A_c \) has no additional singularity. This may be seen as the natural extension of the assumption that \( A_c \) is nonsingular when \( A \) is nonsingular (observe that it is trivially satisfied when \( A_c \) is nonsingular).

**Lemma 3.1.** Let \( A \) be a \( n \times n \) matrix, let \( P \) and \( R \) be full rank matrices of size respectively \( n \times n_c \) and \( n_c \times n \), and let \( A_c = R A P \).
If (3.7) holds, one has, for any pair of matrices $A_c^{(1)}$ and $A_c^{(2)}$ satisfying (3.3),

$$\pi_{R(A^T)}^T P A_c^{(1)} R \pi_{R(A)} = \pi_{R(A^T)}^T P A_c^{(2)} R \pi_{R(A)} ,$$

where $\pi_{R(A^T)}$ and $\pi_{R(A)}$ are projectors onto, respectively, $\mathcal{R}(A^T)$ and $\mathcal{R}(A)$.

Proof. Let $N_{c,R}$ be a $n_c \times k_c$ matrix of rank $k_c$ such that the columns of $P N_{c,R}$ form a basis of $\mathcal{R}(P) \cap \mathcal{N}(A)$. By the condition $\dim(\mathcal{R}(P) \cap \mathcal{N}(A)) = \dim(\mathcal{N}(A_c))$, the columns of $N_{c,R}$ form also a basis of $\mathcal{N}(A_c)$. Since $\mathcal{N}(\pi_{R(A^T)}^T) = \mathcal{R}(\pi_{R(A^T)})^\perp = \mathcal{R}(A^T)^\perp = \mathcal{N}(A)$, one has $\pi_{R(A^T)}^T P N_{c,R} = 0$. Similarly, for $N_{c,L}$ forming a basis of the left null space of $A_c$, one finds that $N_{c,L}^T R \pi_{R(A)} = 0$. Thus, for $i = 1, 2$,

$$\pi_{R(A^T)}^T P A_c^{(i)} R \pi_{R(A)} = \pi_{R(A^T)}^T P (I - N_{c,R}(N_{c,R}^T N_{c,R})^{-1} N_{c,R}^T) A_c^{(i)} (I - N_{c,L}(N_{c,L}^T N_{c,L})^{-1} N_{c,L}^T) R \pi_{R(A)} .$$

The conclusions follows, because, whatever $A_c^{(i)}$ satisfying (3.3), the matrix

$$(I - N_{c,R}(N_{c,R}^T N_{c,R})^{-1} N_{c,R}^T) A_c^{(i)} (I - N_{c,L}(N_{c,L}^T N_{c,L})^{-1} N_{c,L}^T)$$

is the unique generalized inverse of $A_c$ with range $\mathcal{R}(N_{c,R})^\perp$ and null space $\mathcal{R}(N_{c,L})$ [1] Chapter 2, Theorem 10(c)].

In the nonsymmetric case, the assumption (3.7) cannot always be fulfilled in practice. For instance, when solving linear systems associated with Markov chains, the left null space vector is the constant vector which is typically in the range of $R^T$, but the right null space vector is unknown and in general not in the range of $P$. It follows that the coarse grid systems to solve are indeed compatible, but the choice of the generalized inverse has some influence, as the null space component on the coarse grid is prolongated in nontrivial components on the fine grid.

Our main result in the next section uses the assumption (3.7) and can therefore not directly cover such cases. However, it can be applied indirectly via the following Lemma which shows that, if the used generalized inverse has the same rank as $A_c$ (as, e.g., the Moore-Penrose inverse), then the coarse grid correction is in fact identical to that obtained with some truncated prolongation $\tilde{P}$ and restriction $\tilde{R}$ for which the corresponding Galerkin coarse grid matrix $\tilde{R} A \tilde{P}$ is nonsingular.

**Lemma 3.2.** Let $A$ be a $n \times n$ matrix, let $P$ and $R$ be full rank matrices of size respectively $n \times n_c$ and $n_c \times n$, and let $A_c = R A P$.

Let $A_c^g$ be a matrix satisfying (3.3) that has the same rank as $A_c$, and let $V_L, V_R$ be $n_c \times \tilde{n}_c$ matrices whose columns form a basis of $\mathcal{R}(A_c^g)$ and $\mathcal{R}(A_c^{gT})$, respectively, where $\tilde{n}_c$ is the rank of $A_c$.

---

2It is generally observed that the best results are obtained with the Moore-Penrose inverse which provide a minimal norm solution [34, 86], but so far no analysis support this fact.
The \( n \times \tilde{n}_c \) matrix \( \tilde{P} = PV_L \) and the \( \tilde{n}_c \times n \) matrix \( \tilde{R} = V_R^T R \) are such that \( \tilde{R}A\tilde{P} \) is nonsingular and
\[
P A_c^g R = \tilde{P} \left( \tilde{R} A \tilde{P} \right)^{-1} \tilde{R} .
\] (3.8)

Moreover, if
\[
dim(\mathcal{R}(P) \cap \mathcal{N}(A)) = \dim(\mathcal{N}(A_c)) ,
\] (3.9)
there holds
\[
\mathcal{R}(\tilde{P}) + \mathcal{N}(A) = \mathcal{R}(P) + \mathcal{N}(A) ,
\] (3.10)
whereas, if
\[
dim(\mathcal{R}(R^T) \cap \mathcal{N}(A^T)) = \dim(\mathcal{N}(A_c)) ,
\] (3.11)
there holds
\[
\mathcal{R}(\tilde{R}^T) + \mathcal{N}(A^T) = \mathcal{R}(R^T) + \mathcal{N}(A^T) .
\] (3.12)

Proof. One finds \( \tilde{R} A \tilde{P} = V_R^T R A P V_L = V_R^T A_c V_L \). As shown in [1] p. 52, if \( A_c^g \) has the same rank as \( A_c \), then its range is a subspace complementary to \( \mathcal{N}(A_c) \). It follows that for any \( w \neq 0 \), \( A_c V_L w \) is a non trivial vector in \( \mathcal{R}(A_c) \), which further implies that \( V_R^T A_c V_L w \neq 0 \), because otherwise \( A_c V_L w \) would be moreover a nontrivial vector in \( \mathcal{R}(V_R) \cap \mathcal{N}(A_c) \), which is not possible because, as also shown in [1] p. 52, \( \mathcal{N}(A_c) \) is a subspace complementary to \( \mathcal{R}(A_c) \).

Since, as just shown, \( V_R^T A_c V_L w \neq 0 \) for any \( w \neq 0 \), \( \tilde{R} A \tilde{P} = V_R^T A_c V_L \) is nonsingular, which also shows that all three terms \( V_R \), \( A_c \), and \( V_L \) have rank \( \tilde{n}_c \). Then, Theorem 11(d) in Chapter 2 of [1] shows that
\[
A_c^g = V_L (V_R^T A_c V_L)^{-1} V_R^T = V_L \left( \tilde{R} A \tilde{P} \right)^{-1} V_R^T ,
\]
and (3.8) follows straightforwardly.

To show (3.10), let \( P_N \) be a basis of \( \mathcal{R}(P) \cap \mathcal{N}(A) \). One has \( P_N = P N_c \) for some \( n_c \times k_c \) matrix \( N_c \) of rank \( k_c = \dim(\mathcal{N}(A_c)) \). Clearly, all columns of \( N_c \) belong to the null space of \( A_c \), and hence form a basis of that subspace by the condition (3.9). Moreover, we have already seen that \( \mathcal{N}(A_c) = \mathcal{R}(N_c) \) is complementary to \( \mathcal{R}(A_c^g) = \mathcal{R}(V_L) \). Hence the matrix \( V = (V_L \ N_c) \) is \( n_c \times n_c \) and nonsingular. Therefore, the range of \( P \) is equal to that of \( PV \), which is itself equal to \( \mathcal{R}(\tilde{P}) + \mathcal{R}(P_N) \). The equality (3.10) follows (remember that \( \mathcal{R}(P_N) \subset \mathcal{N}(A) \)). The proof of (3.12) is similar. \( \Box \)

### 3.3 Main results for general nonsymmetric matrices

We are now ready to state our generalization of the main results in [27]. The only restrictive assumption is (3.7), which has already been commented in the preceding subsection. Let us add here that, in practice, there are two easy ways to satisfy (3.7): either by enforcing \( \mathcal{N}(A) \subset \mathcal{R}(P) \) and \( \mathcal{N}(A^T) \subset \mathcal{R}(R^T) \) (which is compatible with the popular choice \( R = P^T \) when \( \mathcal{N}(A) = \mathcal{N}(A^T) \)); or by ensuring that \( A_c \) is nonsingular.
On the other hand, when none of these conditions is satisfied, the theorem below is still usable via Lemma 3.2 which allows to substitute the truncated $\hat{P}$ and $\hat{R}$ for the actual $P$ and $R$, and get the same two-level preconditioner with a regular coarse grid matrix. Regarding the stated results, the only difference is that $\hat{P}$ and $\hat{R}$ are then to be such that $\mathcal{R}(\hat{P}) = \mathcal{R}(\hat{P}) + \mathcal{N}(A)$, and $\mathcal{R}(\hat{R}^T) = \mathcal{R}(\hat{R}^T) + \mathcal{N}(A^T)$.

**Theorem 3.3.** Let $A$ be a $n \times n$ matrix, let $M_1$, $M_2$ be $n \times n$ nonsingular matrices such that $(I - M_1^{-1}A)^{\nu_2} (I - M_1^{-1}A)^{\nu_1}$ is the iteration matrix of a convergent iterative method for (1.1). Let $P$ and $R$ be full rank matrices of size respectively $n \times n_c$ and $n_c \times n$, let $A_c = RAP$, and assume that (3.7) holds.

Let $Y$ and $X$ be nonsingular $n \times n$ matrices such that (3.5) and (3.6) hold, and let $\mathcal{B}_{TL}$ be defined by (3.4), where $A_2$ is a matrix satisfying (3.3).

Letting $k = \dim(\mathcal{N}(A))$ and noting $\bar{n}_c$ the rank of $A_c$, let $\hat{P}$ be any $n \times (\bar{n}_c + k)$ matrix and $\hat{R}$ be any $(\bar{n}_c + k) \times n$ matrix such that

$$\mathcal{R}(\hat{P}) = \mathcal{R}(P) + \mathcal{N}(A) \quad \text{and} \quad \mathcal{R}(\hat{R}^T) = \mathcal{R}(\hat{R}^T) + \mathcal{N}(A^T).$$

If $(\hat{P}X\hat{R})$ is nonsingular, the matrix $\mathcal{B}_{TL}A$ has $k$ times the eigenvalue 0 and $\bar{n}_c$ times the eigenvalue 1. The other eigenvalues are the inverses of the $n - \bar{n}_c - k$ nonzero eigenvalues of the generalized eigenvalue problem

$$W_L^T X \left( I - \hat{P} \left( \hat{R} X \hat{P} \right)^{-1} \hat{R} X \right) W_R \mathbf{z} = \mu W_L^T A W_R \mathbf{z},$$

where $W_R$ and $W_L$ are any $n \times (n - k)$ matrices whose columns form a basis of a subspace that is complementary to, respectively, $\mathcal{N}(A)$ and $\mathcal{N}(A^T)$. Moreover, the problem is equivalent to

$$X \left( I - \hat{P} \left( \hat{R} X \hat{P} \right)^{-1} \hat{R} X \right) \mathbf{v} = \mu A \mathbf{v}, \quad \mathbf{v} \in \mathcal{R}(W_R).$$

In addition, for any eigenvalue $\lambda_i \in \sigma(\mathcal{B}_{TL}) \setminus \{0, 1\}$, there exists some vector $\mathbf{z}_i \in \mathbb{C}^n$ such that $\hat{R}\mathbf{z}_i = 0$ and

$$w^T X^{-1} \mathbf{z}_i = \lambda_i w^T A^g \mathbf{z}_i, \quad \forall w \in \mathbb{C}^n : \hat{P}^T w = 0,$$

where $A^g$ is any matrix such that $AA^g A = A$ and rank($A^g$) = rank($A$).

**Proof.** We start with some preliminaries. Observe that, for any $n \times (n - k)$ matrices $V_L$ and $V_R$ whose columns form a basis of respectively $\mathcal{R}(A) = \mathcal{N}(A)^\perp$ and $\mathcal{R}(A^T) = \mathcal{N}(A)^\perp$, there holds that $V_L^T W_L$ and $V_R^T W_R$ are nonsingular: $V_L^T \mathbf{v} = 0$ (resp. $V_R^T \mathbf{v} = 0$) only for $\mathbf{v} \in \mathcal{N}(A^T)$ (resp. $\mathbf{v} \in \mathcal{N}(A)$) whereas, by assumption, $\mathcal{R}(W_L) \cap \mathcal{N}(A^T) = \mathcal{R}(W_R) \cap \mathcal{N}(A) = \{0\}$. Without loss of generality, we can then select such matrices $V_L$ and $V_R$ satisfying in addition

$$V_L^T W_L = W_L^T V_L = V_R^T W_R = W_R^T V_R = I;$$

11
that is, such that $V_L W_L^T$ and $V_R W_R^T$ are projectors onto, respectively, $\mathcal{R}(A)$ and $\mathcal{R}(A^T)$. Note that this implies

$$V_L W_L^T A = A = A W_R V_R^T. \ (3.18)$$

On the other hand, consider $I - X^{-1}A = (I - M_i^{-1}A)^{\nu_1} (I - M_2^{-1}A)^{\nu_2}$. Because it corresponds to a convergent iteration matrix for (1.1), we know by Lemma 2.1 that $X^{-1}A = X^{-1}V_L W_L^T A W_R V_R^T$ has $n - k$ nonzero eigenvalues. By [16, Theorem 1.3.22], this implies that the $(n - k) \times (n - k)$ matrix $(V_R^T X^{-1} V_L)(W_L^T A W_R)$ has only nonzero eigenvalue; i.e., is nonsingular. Hence we may note for further reference that both $(V_R^T X^{-1} V_L)$ and $(W_L^T A W_R)$ are nonsingular $(n - k) \times (n - k)$ matrices.

Next we note that, by Lemma 3.2, one has

$$\mathcal{B}_{TL} = \widetilde{\mathcal{B}}_{TL} = Y^{-1} + (I - M_2^{-1}A)^{\nu_2} \tilde{P} (\tilde{R} A \tilde{P})^{-1} \tilde{R}^T (I - A M_1^{-1})^{\nu_1},$$

where $\widetilde{\mathcal{P}}$ and $\widetilde{R}$ are defined as in Lemma 3.2. With (3.18), we have thus $\mathcal{B}_{TL} A = \widetilde{\mathcal{B}}_{TL} V_L W_L^T A W_R V_R^T$, and, moreover, this latter matrix has, by [16, Theorem 1.3.22], $k$ times the eigenvalue 0 plus the same $n - k$ eigenvalues as $(V_R^T \mathcal{B}_{TL} V_L)(W_L^T A W_R)$. Further, as just seen, $W_L^T A W_R$ is nonsingular, hence we may apply results for the regular case to characterize these $n - k$ eigenvalues. This is the main idea of this proof.

Now, we intend to apply more specifically the results about two-level preconditioning of regular matrices, in particular, Theorem 2.1 of [27]. This is possible because, using (3.18) and $V_R^T W_R = I$, we obtain:

$$I - V_R^T \mathcal{B}_{TL} V_L W_L^T A W_R = V_R^T (I - \widetilde{\mathcal{B}}_{TL} A) W_R$$

$$= V_R^T (I - M_2^{-1}A)^{\nu_2} (I - \widetilde{P} \tilde{A}_c^{-1} \widetilde{R} A) (I - M_1^{-1}A)^{\nu_1} W_R$$

$$= (I - V_R^T M_2^{-1} V_L W_L^T A W_R)^{\nu_2} (I - V_R^T \widetilde{P} \tilde{A}_c^{-1} \widetilde{R} V_L W_L^T A W_R)$$

$$\quad (I - V_R^T M_1^{-1} V_L W_L^T A W_R)^{\nu_1}.$$

Hence, the left hand side matrix is the iteration matrix corresponding to the matrix $W_L^T A W_R$ and the two-level preconditioner associated with the prolongation $V_R^T \tilde{P}$, the restriction $\tilde{R} V_L$, the relaxations operator $V_R^T M_1^{-1} V_L$ and $V_R^T M_2^{-1} V_L$, and the coarse grid matrix

$$\tilde{A}_c = \tilde{R} A \tilde{P} = \tilde{R} V_L W_L^T A W_R V_R^T \tilde{P},$$

which thus corresponds well to a Galerkin projection. Moreover, we know by Lemma 3.2 that $\tilde{A}_c$ is non singular.

Now, Theorem 2.1 of [27] uses the matrix $\mathcal{X}$ such that

$$I - \mathcal{X}^{-1} W_L^T A W_R = (I - V_R^T M_1^{-1} V_L W_L^T A W_R)^{\nu_2} (I - V_R^T M_2^{-1} V_L W_L^T A W_R)^{\nu_2},$$

and requires that

$$X_c = \widetilde{R} V_L \mathcal{X} V_R^T \tilde{P}$$

is nonsingular (for the case where the restriction is the transpose of the prolongation, this condition has been further shown in [29] to be the necessary and sufficient condition for
the nonsingularity of the two-level preconditioning matrix). To investigate this condition, observe that

\[
I - \overline{X}^{-1} W_L^T A W_R = V_R^T (I - M_1^{-1} A) \nu_1 (I - M_2^{-1} A) \nu_2 W_R = V_R^T (I - X^{-1} A) W_R = I - V_R^T X^{-1} V_L W_L^T A W_R;
\]

that is,

\[
\overline{X}^{-1} = V_R^T X^{-1} V_L
\]

(Remember that we showed above that \(V_R^T X^{-1} V_L\) is nonsingular, hence expressing it as \(\overline{X}^{-1}\) is not an abuse of notation). Further, note that \(V_L (V_R^T X^{-1} V_L)^{-1} V_R^T X^{-1}\) is a projector. Its right kernel is the set of vectors which, multiplied by \(X\), are orthogonal to \(\mathcal{R}(V_R) = \mathcal{R}(A^T) = \mathcal{N}(A)^\perp\); i.e., a basis of the right kernel is formed by the columns of \(X N_R\), where \(N_R\) is a \(n \times k\) whose columns span a basis of \(\mathcal{N}(A)\). On the other hand, the left kernel of \(V_L (V_R^T X^{-1} V_L)^{-1} V_R^T X^{-1}\) is the set of vector orthogonal to \(\mathcal{R}(V_L) = \mathcal{R}(A) = \mathcal{N}(A^T)^\perp\); i.e., a basis of the left kernel is obtained taking the columns of a \(n \times k\) matrix \(N_L\) that form a basis of \(\mathcal{N}(A^T)\). Noting that the nonsingularity of \(\tilde{R} X \tilde{P}\) implies that of \(N_R^T X N_L\) (because \(\mathcal{R}(\tilde{R}^T)\) contains \(\mathcal{R}(N_R)\) and \(\mathcal{R}(\tilde{P})\) contains \(\mathcal{R}(N_L)\)), we have thus proved that

\[
V_L \overline{X} V_R^T X^{-1} = V_L (V_R^T X^{-1} V_L)^{-1} V_R^T X^{-1} = I - X N_R (N_L^T X N_R)^{-1} N_L^T,
\]

and hence that

\[
X_c = \tilde{R} X \tilde{P} - \tilde{R} X N_R (N_L^T X N_R)^{-1} N_L^T X \tilde{P}.
\]

Here it is worth noting that, setting

\[
\mathcal{T} = \begin{pmatrix} \tilde{P} & N_R \end{pmatrix} \text{ and } \mathcal{R} = \begin{pmatrix} \tilde{R} \\ N_L^T \end{pmatrix}
\]

\(X_c\) coincides with the Schur complement obtained by eliminating the bottom right block in

\[
\overline{R} X \mathcal{T} = \begin{pmatrix} \tilde{R} X \tilde{P} & \tilde{R} X N_R \\ N_L^T X \tilde{P} & N_L^T X N_R \end{pmatrix}.
\]

(3.20)

Moreover, \(\mathcal{T}\) is full rank and, since (3.10) applies ((3.9) holds by (3.7)), has range \(\mathcal{R}(\tilde{P}) + \mathcal{N}(A) = \mathcal{R}(P) + \mathcal{N}(A) = \mathcal{R}(\tilde{P})\). Hence there exists a nonsingular matrix \(G_P\) such that \(\tilde{P} = \mathcal{T} G_P\). Similarly, \(\overline{R}\) is full rank and such that \(\tilde{R} = G_R \mathcal{T}\) for some nonsingular \(G_R\). Therefore,

\[
\mathcal{R} X \mathcal{T} = G_R \tilde{R} X \tilde{P} G_P,
\]

and it follows that the assumed nonsingularity of \(\tilde{R} X \tilde{P}\) implies that of \(\overline{R} X \mathcal{T}\), and hence that of its Schur complement \(X_c\).

We can thus use Theorem 2.1 of [27] to characterize the eigenvalues of \(V_R^T \tilde{B}_{TL} V_L W_L^T A W_R\). Remembering that, in the present context, the prolongation is \(V_R^T \tilde{P}\), the restriction is
that the matrix $\tilde{R} V_L$, and that the matrix $\tilde{X}$ as defined above plays the role of $X$, this result tells us that $V_R^T B_{TL} V_L W_L^T A W_R$ (and therefore $B_{TL} A$) has exactly $n - k$ nonzero eigenvalues, among which $\tilde{n}_c$ (the column size of $V_R^T \tilde{P}$ and $V_L^T \tilde{R} F$) are equal to 1, the remaining $n - \tilde{n}_c - k$ eigenvalue being the inverse of the nonzero eigenvalues of the generalized eigenvalue problem

$$\tilde{X} \left( I - V_R^T \tilde{P} X_c^{-1} \tilde{R} V_L \tilde{X} \right) \mathbf{w} = \mu W_L^T A W_R \mathbf{w} . \quad (3.21)$$

We now conclude the proof of the statement related to (3.14) by showing that the matrix in its left hand sides is in fact equal to the matrix in the left hand side of (3.21). To this aim, remember that $\tilde{P} = \tilde{P} G_P$ and $\tilde{R} = G_R \tilde{R}$ hold for some nonsingular $G_P$ and $G_R$. Hence, noting $*$ matrix blocks whose expression is unimportant, and remembering that $X_c$ is a Schur complement in (3.20),

$$V_R^T \tilde{P} \left( \tilde{R} X \tilde{P} \right)^{-1} \tilde{R} V_L = V_R^T \tilde{P} (R X \tilde{P})^{-1} \tilde{R} V_L$$

$$= V_R^T \left( \tilde{P} \ N_R \right) \left( \tilde{R} X \tilde{P} \ N_L X \tilde{P} \ N_L^T X \tilde{N}_R \right)^{-1} \left( \tilde{R} \ N_L^T \right) V_L$$

$$= \left( V_R^T \tilde{P} \ V_R^T N_R \right) \left( \begin{array}{cc} X_c^{-1} & * \\ * & * \end{array} \right) \left( \tilde{R} V_L \ N_L^T V_L \right)$$

$$= V_R^T \tilde{P} X_c^{-1} \tilde{R} V_L ,$$

the last equality being obtained because $V_R^T N_R = 0 (\mathcal{R}(A^T) \perp \mathcal{N}(A))$ and $N_L^T V_L = 0 (\mathcal{R}(A) \perp \mathcal{N}(A^T))$. Therefore, using the notation $\pi_N = N_R (N_L X N_R)^{-1} N_L^T X$, $\hat{\pi} = \tilde{P} \left( \tilde{R} X \tilde{P} \right)^{-1} \tilde{R} X$, we find, using also (3.17) and $V_L \tilde{X} V_R^T = X (I - \pi_N)$ (from (3.19)):

$$\tilde{X} \left( I - V_R^T \tilde{P} X_c^{-1} \tilde{R} V_L \tilde{X} \right) = W_L^T V_L \tilde{X} \left( I - V_R^T \tilde{P} \left( \tilde{R} X \tilde{P} \right)^{-1} \tilde{R} V_L \tilde{X} \right) V_R^T W_R$$

$$= W_L^T (V_L \tilde{X} V_R^T) \left( I - \hat{\pi} \left( \tilde{R} X \tilde{P} \right)^{-1} \tilde{R} \left( V_L \tilde{X} V_R^T \right) \right) W_R$$

$$= W_L^T X (I - \pi_N) (I - \pi_N) W_R$$

$$= W_L^T X \left( I - \pi_N \right) (I - \pi_N) \hat{\pi} \pi_N W_R$$

$$= W_L^T X \left( I - \hat{\pi} \right) W_R ,$$

the last equality following from the fact that $\hat{\pi}$ and $\pi_N$ are projectors satisfying $\mathcal{N}(\hat{\pi}) = \mathcal{R} (I - \hat{\pi}) \subset \mathcal{N}(\pi_N)$ and $\mathcal{R}(\pi_N) \subset \mathcal{R}(\hat{\pi})$, hence $\pi_N (I - \hat{\pi}) = 0$ and $\hat{\pi} \pi_N = \pi_N$.

On the other hand, multiplying both sides of (3.14) to the left by $V_L$ yields (3.15) because of (3.18), whereas $\mathcal{N}(A^T) = \mathcal{R}(N_L) \subset \mathcal{R}(\tilde{R} F)$ implies $N_L^T X (I - \hat{\pi}) = 0$, showing that the range of $X (I - \hat{\pi}) = 0$ is a subset of $\mathcal{R}(A) = \mathcal{R}(V_L W_L^T)$, implying $V_L W_L^T X (I - \hat{\pi}) = X (I - \hat{\pi})$. As, conversely, (3.15) straightforwardly implies (3.14), this concludes the proof of the statement related to (3.15).
Finally, to prove (3.16), note first that the condition rank($A^g$) = rank($A$) ensures that $R(A^g)$ is complementary to $N(A)$ [1, p. 52]; hence we may select $R(W_R) = R(A^g)$. Consider then a particular solution $(\mu_i, v_i)$ of (3.15). Because $R(W_R) = R(A^g)$, $v_i = A^g z_i$ for some $z_i \in R(A)$. Since $AA^g$ in a projector onto $R(A)$ [1, p. 52], (3.15) yields

$$X \left( I - \hat{P} \left( \hat{R} X \hat{P} \right)^{-1} \hat{R} X \right) A^g z_i = \mu_i z_i.$$  

Multiplying both sides to the left by $\hat{R}$ shows then that $\hat{R} z_i = 0$, whereas multiplying both sides to the left by $w^T X^{-1}$ yields (3.16).  

It is worth noting that the above theorem can be applied to a nonsingular matrix $A$. Then, the main assumption (3.7) amounts in fact to dim($N(A_c)$) = 0, i.e., to assume that $A_c$ is nonsingular, and hence $A_c^{-1}$. Moreover, (3.13) is most easily satisfied using $\hat{P} = P$, $\hat{R} = R$, whereas one can choose $W_R = W_L = I$. With these simplifications, the statements in Theorem 3.3 in fact just reproduce the main statements in Theorem 2.1 of  

[27], which are thus properly generalized by the new formulation.

On the other hand, when applied to singular $A$, Theorem 3.3 first tells us that the algebraic multiplicity of the eigenvalue 0 of the preconditioned matrix is equal to dim($N(A)$). Hence, Lemma 2.2 can be applied to analyze GMRES iterations, whereas condition (1) of Lemma 2.1 holds, and hence the convergence of stationary iterations is guaranteed if the nonzero eigenvalues $\lambda$ of $BA$ satisfy $|1 - \lambda| < 1$, which holds if and only if the nonzero eigenvalues $\mu$ of (3.14) or (3.15) satisfy $|1 - \mu^{-1}| < 1$.

### 3.4 The nonsymmetric semidefinite case

Besides the classical case where the system matrix is symmetric and positive semidefinite (considered in the next subsection), more insight can be gained for nonsymmetric matrices which are positive (semi)definite in $\mathbb{R}^n$; that is, for matrices $A$ such that

$$v^T A v \geq 0 \quad \forall v \in \mathbb{R}^n,$$

or, equivalently, such that the symmetric part $A = \frac{1}{2}(A + A^T)$ is positive (semi)definite. For regular matrices, this case is discussed in [27, 29], and we want to extend here these results to the singular case, taking benefit from the fact that they are corollaries of the general result for nonsymmetric matrices considered in the preceding subsection.

Note that with (3.22), $v^T (A + A^T) v = 0$ is possible only for null space vectors of $A + A^T$. This implies that the left and right null space of $A$ coincide, and thus further coincide with the null space of $A + A^T$:

$$A v = 0 \Rightarrow v^T (A + A^T) v = 0 \Rightarrow (A + A^T) v = 0 \Rightarrow A^T v = 0.$$  

Thus, if $R = P^T$, one may always select $\hat{R}$ such that $\hat{R} = \hat{P}^T$.

Now, to obtain a clear localization of the eigenvalues from Theorem 3.3, we need not only to assume (3.22) and $R = P^T$, but also that $X$ is SPD. This latter assumption is
particularly restrictive and means that, in practice, the analysis is in general possible only for a simplified two-level scheme using only a single (either pre- or post-) smoothing step with a simple smoother such as damped Jacobi (then $X$ is equal to either $M_1$ or $M_2$).

Altogether, the additional assumptions used in the following theorem are thus

$$A$$ is positive (semi)definite in $\mathbb{R}^n$, $R = P^T$, and $X$ is SPD. \hfill (3.23)

**Theorem 3.4.** Let the assumption of Theorem 3.3 hold, assume in addition (3.23), and select $\hat{R} = \hat{P}^T$. Then, $B_{TL} A$ has $k$ times the eigenvalue 0, $\tilde{n}_c$ times the eigenvalue 1, and the $n - \tilde{n}_c - k$ other eigenvalues $\lambda$ satisfy

$$|\lambda - \alpha_X| \leq \alpha_X,$$ \hfill (3.24)

$$\Re(\lambda) \geq K_X^{-1},$$ \hfill (3.25)

where, for any matrix $A^g$ such that, $A A^g A = A$ and $\text{rank}(A^g) = \text{rank}(A)$,

$$\alpha_X = \sup_{\begin{smallmatrix} \nu \in \mathbb{R}^n, \nu \notin \mathcal{N}(A) \\ \hat{P}^T \nu = 0 \end{smallmatrix}} \frac{\nu^T A^T X^{-1} A \nu}{2 \nu^T A \nu} = \sup_{\begin{smallmatrix} \nu \in \mathbb{R}^n \\ \hat{P}^T \nu = 0 \end{smallmatrix}} \frac{\nu^T X^{-1} \nu}{2 \nu^T A^g \nu},$$ \hfill (3.26)

$$K_X = \sup_{\begin{smallmatrix} \nu \in \mathbb{R}^n \\ \nu \notin \mathcal{N}(A) \end{smallmatrix}} \nu^T X \left( I - \hat{P} \left( \hat{P}^T X \hat{P} \right)^{-1} \hat{P}^T X \right) \nu / \nu^T A \nu.$$ \hfill (3.27)

Moreover, if, for some number $\alpha$ there holds

$$\| (\alpha I - X^{-1} A) \nu \|_X \leq \alpha \| \nu \|_X \quad \forall \nu : \hat{P}^T A \nu = 0,$$ \hfill (3.28)

then $\alpha_X \leq \alpha$.

**Proof.** Exploiting (3.15), the proof of (3.25) given in Corollary 2.2 of [27] applies verbatim to the singular case as well.

Consider next the Moore-Penrose inverse $A^+$. Note that $A A^+$ is an orthogonal projector onto the range of $A$ and $A^+ A$ an orthogonal projector onto the range of $A^T$ which, in addition, coincides with that of $A$ (since $\mathcal{N}(A) = \mathcal{N}(A^T)$), and thus also coincides with the range of both $A^+$ and $(A^+)^T$. Then:

$$(A^+)^T (A + A^T)(A^+) = (A^+)^T (A A^+) + (A A^+)^T A^+ = (A^+)^T + A^+,$$

which shows that the positive (semi)definiteness of $A$ entails that of $A^+$; i.e., $\nu^T A^+ \nu \geq 0$ for all $\nu$. This allows to exploit (3.16) with $\nu = z_i$ exactly as in the proof of Corollary 2.1 of [27] to show that (3.24) holds with $\alpha_X$ as in the right hand side of (3.26) with $A^g = A^+$. Further, the condition $P^T \nu = 0$ implies that $\nu \in \mathcal{R}(A)$ (Remember that $\mathcal{R}(\hat{P})$ contains $\mathcal{N}(A^T) = \mathcal{R}(A)^\perp$); hence $\nu = \pi_{\mathcal{R}(A)} \nu$, where $\pi_{\mathcal{R}(A)}$ is the orthogonal projector onto the range of $A$. Therefore, since $A^+ = \pi_{\mathcal{R}(A)} A^g \pi_{\mathcal{R}(A)}$ for any $A^g$ [1], Chapter 2, Theorem 10(c)],
one has, for any such \( \mathbf{v} \), \( \mathbf{v}^T A^\# \mathbf{v} = \mathbf{v}^T \pi_{R(A)} A^\# \pi_{R(A)} \mathbf{v} = \mathbf{v}^T A^+ \mathbf{v} \). Moreover, by setting \( \mathbf{w} = A \mathbf{v} \) in the right hand side of (3.26) (remember that \( \bar{P}^T \mathbf{v} = 0 \) implies \( \mathbf{v} \in \mathcal{R}(A) \)), one can check the equality with the middle term.

Finally,

\[
\frac{\mathbf{w}^T A^T (X^{-1}) A \mathbf{w}}{\mathbf{w}^T A \mathbf{w}} \leq 2 \alpha \iff \mathbf{w}^T A^T X^{-1} A \mathbf{w} \geq \alpha \mathbf{w}^T (A + A^T) \mathbf{w} \iff \mathbf{w}^T ((\alpha I - A^T X^{-1}) X (\alpha I - X^{-1} A) - \alpha^2 X) \mathbf{w} \leq 0 \iff \|((\alpha I - X^{-1} A) \mathbf{w})_X \| \leq \alpha \|\mathbf{w}\|_X ,
\]

showing that (3.28) implies \( \alpha_X \leq \alpha \) as claimed. ■

3.5 The symmetric semidefinite case

Here we address the generalization of the results for SPD matrices originally obtained in [4, 6, 13, 14, 30, 32]. We use the fact that these results can be recovered as corollaries of the analysis for general nonsymmetric matrices considered in Section 3.3, even though the original proofs were quite different. More precisely, this can be done in two steps. Firstly, the sharp estimate in [13, 14] can be obtained by particularizing to the symmetric case the results in [27, Theorem 2.1]. Next, as also discussed in [13, 14], the previous (non sharp) estimates can be derived by combining the sharp bound with some further inequalities. We accordingly proceed in two steps here: Theorem 3.5 below extends the bound in [13, 14] to symmetric semidefinite matrices, whereas Theorem 3.6 generalizes the inequalities that allow to deduce more tractable bounds, in particular those appeared earlier in [4, 6, 30, 32].

To make things clear, the assumption used in this subsection are:

\[ A \text{ is symmetric and (semi)definite, } R = P^T, \ Y \text{ and } X \text{ are SPD} \, . \quad (3.29) \]

Then, the two potentially restrictive assumptions of Theorem 3.3 are automatically satisfied: on the one hand, \( \bar{P}^T X \bar{P} \) is always non singular for SPD \( X \), and, on the other hand, (3.7) also always holds: because \( A \) semidefinite implies that \( A_c \) is semidefinite as well, one has, for any \( \mathbf{v}_c \),

\[ A_c \mathbf{v}_c = 0 \iff \mathbf{v}_c^T (P^T A P) \mathbf{v}_c = 0 \iff (P \mathbf{v}_c)^T A (P \mathbf{v}_c) = 0 \iff A (P \mathbf{v}_c) = 0 , \]

showing that any vector in the null space of \( A_c \) corresponds to a vector in \( \mathcal{R}(P) \cap \mathcal{N}(A) \).

Before going further, it is worth discussing how to make sure that \( Y \) and \( X \) are SPD. The easiest way is to use only one single pre- or post-smoothing step (i.e., either \( \nu_1 = 1 \), \( \nu_2 = 0 \) or \( \nu_1 = 0 \), \( \nu_2 = 1 \)) with a SPD preconditioner \( M_1 \) or \( M_2 \). A more standard option is to use \( \nu_1 = \nu_2 = 1 \) and \( M_1 = M_2^T = M \) such that

\[ M + M^T - A \text{ is SPD} \, . \quad (3.30) \]

Then, see [2, 8, 19], the iteration matrix \( (I - M_{-T}) (I - M^{-1} A) \) is always convergent and

\[ Y = M (M + M^T - A) M^T , \quad X = M^T (M + M^T - A) M \]
are valid choices. Moreover, one has

\[ 0 \leq \lambda \leq 1 \quad \forall \lambda \in \sigma(Y^{-1}A) = \sigma(X^{-1}A) \, . \tag{3.31} \]

On the other hand, see (3.4), in such cases, the two-level preconditioning matrix \( B_{TL} \) is SPD and, hence, can be used in combination with the conjugate gradient method (see Lemma 2.5).

Two important particular cases satisfying (3.30) are, on the one hand, when \( M \) is itself SPD and such that \( I - M^{-1}A \) is convergent, and, on the other hand, symmetrized Gauss–Seidel smoothing, for which \( M_1 = \text{upp}(A) \), \( M_2 = \text{low}(A) \) and therefore \( M_1 + M_2 - A = \text{diag}(A) \) is SPD as soon as all diagonal entries of \( A \) are positive.

Now, our generalization of the sharp bounds for the SPD case can be stated as follows.

**Theorem 3.5.** Let the assumption of Theorem 3.3 hold, assume in addition (3.29), and select \( \hat{R} = \hat{P}^T \). Then, \( B_{TL}A \) has \( k \) times the eigenvalue 0 and \( n - k \) positive eigenvalues satisfying

\[
\max_{\lambda \in \sigma(B_{TL}A)} \lambda \leq \max \left( 1, \max_{\lambda \in \sigma(Y^{-1}A)} \lambda \right) , \tag{3.32}
\]

\[
\min_{\lambda \in \sigma(B_{TL}A)} \lambda = \min (1, K_X^{-1}) , \tag{3.33}
\]

where

\[
K_X = \sup_{v \in \mathbb{R}^n \setminus N(A)} v^T X \left( I - \hat{P} (\hat{P}^T \hat{P})^{-1} \hat{P}^T X \right) v \div v^T A v . \tag{3.34}
\]

In particular, if (3.31) holds,

\[
\max_{\lambda \in \sigma(B_{TL}A) \setminus \lambda \neq 1} \lambda = 1 , \tag{3.35}
\]

\[
\kappa_{eff} = \frac{\max_{\lambda \in \sigma(B_{TL}A)}}{\min_{\lambda \in \sigma(B_{TL}A) \setminus \lambda \neq 0}} \lambda = K_X , \tag{3.36}
\]

\[
\rho_{eff} = \min_{\lambda \in \sigma(B_{TL}A) \setminus \lambda \neq 0} |1 - \lambda| = 1 - K_X^{-1} . \tag{3.37}
\]

**Proof.** The eigenvalue estimates are straightforward corollaries of (3.14) or (3.15) under the given additional conditions. \[Q.E.D.\]

The matrix in the numerator of \( K_X \) has often a complicated form, except if one deliberately chooses to analyze a simplified scheme with, say, only a single smoothing step. This matrix can be seen as the product of \( X \) with projectors: letting \( \hat{\pi} = I - \hat{P} (\hat{P}^T \hat{P})^{-1} \hat{P}^T \), one has \( X(I - \hat{P} (\hat{P}^T \hat{P})^{-1} \hat{P}^T X) = \hat{\pi}^T X \hat{\pi} \). The idea behind more tractable estimates (including earlier bounds) is to substitute a simpler matrix for \( X \), and/or to use simpler projectors. Of course, inequalities are then needed to deduce eigenvalue bounds from the associated quantities. These are proved in the following theorem.
Theorem 3.6. Let the assumption of Theorem 3.3 hold, assume in addition (3.29), and select \( \hat{R} = \hat{P}^T \). The quantity

\[
K_X = \sup_{v \in \mathbb{R}^n \setminus N(A)} \frac{v^T X \left( I - \hat{P}(\hat{P}^T X \hat{P})^{-1} \hat{P}^T X \right) v}{v^T A v},
\]

is also the smallest constant \( K \) such that

\[
\|A(I - P A_c^g P^T A)v\|_{X^{-1}}^2 \geq K^{-1} \| (I - P A_c^g P^T A)v\|_A^2 \quad \forall v \in \mathcal{R}(A). \tag{3.39}
\]

On the other hand, letting, for any projector \( \pi \) with null space equal to \( \mathcal{R}(\hat{P}) \),

\[
K_X(\pi) = \sup_{v \in \mathbb{R}^n \setminus N(A)} \frac{v^T \pi^T X \pi v}{v^T A v},
\]

there holds

\[
K_X = \min_{\pi: \text{projector with } N(\pi) = \mathcal{R}(\hat{P})} K_X(\pi) \tag{3.41}
\]

with \( \arg \min_{\pi} = I - \hat{P}(\hat{P}^T X \hat{P})^{-1} \hat{P}^T X \). Moreover,

\[
\|\pi\|_X^2 \inf_{v \in \mathcal{N}(A)} \frac{v^T X v}{v^T A v} \leq K_X(\pi) \leq K_X \|\pi\|_X^2. \tag{3.42}
\]

Further, let \( \delta_X \) be the largest constant \( \delta \) such that

\[
\|A v\|_{X^{-1}}^2 \geq \delta \| (I - P A_c^g P^T A)v\|_A^2 \quad \forall v \in \mathcal{R}(A). \tag{3.43}
\]

Setting

\[
\tilde{\pi} = \left( I - N(N^T X N)^{-1} N^T X \right) (I - P A_c^g P^T A) \tag{3.44}
\]

where \( N \) is any \( n \times k \) matrix whose columns form a basis of \( \mathcal{N}(A) \), there holds

\[
\delta_X = (K_X(\tilde{\pi}))^{-1}. \tag{3.45}
\]

In addition, if \( X = M^T (M + M^T - A)^{-1} M \) for some non singular matrix \( M \), \( \delta_X \) is also the largest constant \( \delta \) such that

\[
\|(I - M^{-T} A) v\|_A^2 \leq \|v\|_A^2 - \delta \| (I - P A_c^g P^T A)v\|_A^2 \quad \forall v \in \mathbb{R}^n. \tag{3.46}
\]

Finally, for any \( n \times n \) SPD matrices \( G \), defining \( K_G \) as in (3.38) with \( G \) instead of \( X \), there holds

\[
\min_{v \neq 0} \frac{v^T X v}{v^T G v} K_G \leq K_X \leq \max_{v \neq 0} \frac{v^T X v}{v^T G v} K_G. \tag{3.47}
\]
Proof. We start with some preliminaries. Since $A_c A_c^p$ is a projector onto $\mathcal{R}(A_c)$ [11 p. 52], letting $N_c$ be a matrix whose columns form a basis of $\mathcal{N}(A_c)$, we have $A_c A_c^p = I - E_c N_c^T$ for some $E_c$. Similarly, $A_c A_c^T = (A_c^p A_c^p)^T (A_c$ is symmetric but $A_c^p$ may be unsymmetric) is a projector onto $\mathcal{R}(A_c)$ and hence $A_c A_c^T = I - F_c N_c^T$ for some $F_c$. Letting $Q = P A_c^p P^T A$, one has then $A(I - Q) = A P(I - A_c^p A_c) = A P N_c F_c^T = 0$ (remember that $\mathcal{R}(P N_c) \subset \mathcal{N}(A)$). Similarly, one finds $P^T A(I - Q) = (I - A_c A_c^p)^T P^T A = 0$ and $P^T (I - Q^T) A = 0$, and one may check that $Q^T A Q = A P A_c^T A_c A_c^T P^T A = A P (A_c A_c^T) A_c^T P^T A = A Q$, which further implies $(I - Q^T) A(I - Q) = (I - Q^T) A$.

We now prove (3.39). To this aim, first note that if $X = \alpha \tilde{X}$ for some positive $\alpha$, then $K_x = \alpha K_{\tilde{X}}$, whereas the smallest admissible $K$ in (3.39) also scales linearly with $\alpha$. Hence we may assume without loss of generality that $\tilde{X}$ is scaled in such a way that $\max_{\lambda \in \sigma(\tilde{X}^{-1} A)} \lambda \leq 1$, which implies $K_{\tilde{X}} \leq 1$. Consider then the generalized eigenvalue problem

$$(I - Q^T) A X^{-1} A(I - Q) z = \mu A z, \quad z \in \mathcal{R}(A).$$

(3.48)

Because $A$ is SPD on $\mathcal{R}(A)$ and the left hand side matrix is symmetric nonnegative definite, the eigenvalues are nonnegative and there exists a set of eigenvectors $\{z_i\}$ forming a basis of $\mathcal{R}(A)$ and further satisfying $z_i^T A z_j = \delta_{ij}$. Moreover, multiplying both sides to the left by $P^T$, one sees that $P^T A z_i = 0$ for all eigenvectors associated with positive eigenvalues (as seen above, $P^T (I - Q^T) A = 0$). On the other hand, the eigenvectors $z_i$ associated with the eigenvalue 0 are such that $(I - Q) z_i \in \mathcal{N}(A)$. Then expand $v \in \mathcal{R}(A)$ on this eigenvector basis: $v = \sum_i \alpha_i z_i$. Using the just seen properties of $z_i$, one finds

$$\|A(I - P A_c^p P^T A) v\|_X^2 = \sum_{i: \mu_i > 0} |\alpha_i|^2 \mu_i \quad \text{and} \quad \|(I - P A_c^p P^T A) v\|_A^2 = \sum_{i: \mu_i > 0} |\alpha_i|^2.$$  

It follows that the smallest possible $K$ in (3.39) is the inverse of the smallest nonzero eigenvalue of (3.48), $\mu_{\text{min}}$, and one may further note that $\mu_{\text{min}} \leq \max_{\lambda \in \sigma(\tilde{X}^{-1} A)} \lambda \leq 1$.

On the other hand, let $z_i$ be any eigenvector of (3.48) associated with a positive eigenvalue $\mu_i$. Since, by the condition $P^T A z_i = 0$, the equation (3.48) implies

$$A z_i - (I - Q^T) A X^{-1} A z_i = (I - Q^T) (I - A X^{-1}) (A z_i) = (1 - \mu_i) (A z_i),$$

one sees that $x_i = A z_i$ is an eigenvector of the matrix

$$Z = (I - Q^T) (I - A X^{-1})$$

associated with the eigenvalue $1 - \mu_i$. Conversely, since $Z$ has $\mathcal{N}(A)$ as left eigenspace associated with the eigenvalue 1, all its (right) eigenvectors $x_i$ associated with eigenvalues different from 1 are orthogonal to $\mathcal{N}(A)$; i.e., they belong to the range of $A$ and can thus be written $x_i = A z_i$ for $z_i \in \mathcal{R}(A)$. Since $P^T Z A = 0$, such $z_i$ satisfy $P^T A z_i = 0$ when the eigenvalue $\lambda_i$ is nonzero, and are therefore eigenvectors of (3.48) with eigenvalue $\mu_i = 1 - \lambda_i$. Below we show that $Z$ has further the same eigenvalues as the iteration matrix $I - B_{\text{TFF}} A$, and hence that these $\mu_i = 1 - \lambda_i$ coincide with the eigenvalues of $B_{\text{TFF}} A$ that are different from 0 and 1. Using $K_{\tilde{X}} \leq 1$ and (3.33) of Theorem 3.5 proves then the required
result. To conclude this part of the proof, we now show that $Z$ has the same eigenvalues as $I - B_T A$: $Z$ has the same eigenvalues as its transpose

$$Z^T = (I - X^{-1}A)(I - Q) = (I - M_1^{-1}A)^{\nu_1} (I - M_2^{-1}A)^{\nu_2} (I - Q)$$

which, by [16, Theorem 1.3.22], has the same spectrum as $I - B_T A$.

Next, regarding (3.41) and (3.47), these are standard results for the regular case; see [13, 14, 22, 29]. We refer the reader to these works because the related developments carry straightforwardly over the singular case, since they are based on inequalities involving only the numerators of the expressions defining $K_X$ and $K_X(\pi)$. Similarly, the equivalence between (3.43) and (3.46) is based on $\|A\|_X^2 = v^T A(M^{-1} + M^{-T} - M^{-1}AM^{-T})A v = v^T (A - (I - A^{-1}AM^{-T}))(I - M^{-T}A) v$, which raises no particular comment for the singular case.

Regarding (3.42), note that

$$\|\pi\|_X^2 = \sup_{v \in \mathbb{R}^n} \frac{v^T \pi^T X \pi v}{v^T X v}.$$

The left inequality immediately follows, whereas noting $\pi_X = I - \hat{P}(\hat{P}^T X \hat{P})^{-1} \hat{P}^T X$, one has $\pi \pi_X = \pi$, and hence

$$v^T \pi^T X \pi v = v^T \pi_X^T \pi^T X \pi \pi_X v \leq \|\pi\|_X^2 v^T \pi_X^T X \pi_X v,$$

from which we deduce the right inequality.

It remains to prove (3.45). Remembering that $(I - Q^T) A(I - Q) = (I - Q^T) A$ (see the beginning of this proof), one sees that the largest possible $\delta$ in (3.43) is the inverse of the largest eigenvalue of the generalized eigenvalue problem

$$(I - Q^T) A = \mu A X^{-1} A v, \quad v \in W,$$

where $W$ is any fixed subspace complementary to $N(A)$. On the other hand, the right hand side of (3.40) with $\pi = \bar{\pi}$ is the largest eigenvalue of

$$(I - Q^T) X (I - N(N^T X N)^{-1} N^T X) (I - Q) v = \mu A v, \quad v \in W. \quad (3.50)$$

Next, $R(X^{-1} A)$ is complementary to $N(A)$ by Lemma 2.1, which applies because $I - X^{-1} A$ corresponds to a convergent iteration by assumption. We can thus select $W = R(X^{-1} A)$, and replace $v$ in (3.50) by $X^{-1} A w$, where $w$ is again a vector in $W$; that is, (3.50) is equivalent to

$$(I - Q^T) X (I - N(N^T X N)^{-1} N^T X) (I - Q) X^{-1} A w = \mu A X^{-1} A w, \quad v \in W. \quad (3.51)$$

Now, multiplying both sides of (3.51) to the left by $P^T$, the left hand side is zero because, as seen at the beginning of this proof, $P^T(I - Q^T) = F_c N_c^T P^T$ with $P N_c \subset N(A)$. Hence any
eigenvector corresponding to a nonzero eigenvalue satisfies \( P^T A X^{-1} A v = 0 \). Similarly, any eigenvector of \( (3.49) \) corresponding to a nonzero eigenvalue satisfies \( P^T A X^{-1} A w = 0 \). Further, when \( P^T A X^{-1} A w = 0 \) holds, \( (I - Q) X^{-1} A w = (I - P A^g P^T A) X^{-1} A w = X^{-1} A w \) and \( (3.51) \) amounts to

\[
(I - Q^T) X (I - N(NT X N)^{-1}NT X) (I - P A^g P^T A) X^{-1} A w \\
= (I - Q^T) (I - X N(NT X N)^{-1}NT) A w \\
= (I - Q^T) A w ,
\]

showing that the problems \( (3.49) \) and \( (3.51) \) have indeed the same nonzero eigenvalues and associated eigenvectors.

McCormick’s bound [23] (extended to the singular case in [3]) amounts to analyze the constant \( \delta \) in either \( (3.43) \) or \( (3.46) \). The inequality \( K_X \leq \delta X^{-1} \) (that can be deduced by combining \( (3.45) \) with \( (3.41) \)) extends here to the singular case of a well known fact, whereas \( (3.45) \) and \( (3.42) \) (applied to \( \tilde{\pi} \)), that originate from [24, 25], allow to see more closely the relationship between this bound and sharp two-level estimates.

Regarding the practical exploitation of these results, the singular case does not raise particular comments, and hence we refer the reader to the literature. For instance, a popular approach amounts to analyze \( K_D \) instead of \( K_X \), where \( D = \text{diag}(A) \). The analysis of the constant in the right inequality \( (3.47) \) is then sometimes referred to as smoothing property analysis [30, 32], and available results straightforwardly carry over the singular case, since both \( X \) and \( D \) are SPD, anyway. For instance, for symmetric M-matrices, it can be shown that this constant is at most 4 when \( X \) correspond to symmetrized Gauss–Seidel smoothing (i.e., forward Gauss–Seidel as pre-smoother and backward Gauss-Seidel as post-smoother); see, e.g., Theorem A.3.1 in [32] and the subsequent discussion, which can be applied verbatim to singular M-matrices as well.

**Application to classical AMG**

We now explore how the above theorems can be used to extend to singular systems the classical two-level convergence estimate for Ruge-Stüben AMG methods [5, 30, 32]. This analysis applies when the system matrix \( A \) is a symmetric M-matrices with nonnegative row sum, hence we consider here symmetric M-matrices with zero row sum (singular M-matrices with nonnegative row sum have always zero row sum). We further assume that \( A \) is irreducible, implying that the null space has dimension 1 and is thus spanned by the constant vector.

Now, the convergence proof is based on the approach just mentioned, where one combines an analysis of \( K_D \) with the smoothing property analysis. According to the above discussion, this latter does not raise particular comments in the singular case, hence we focus here on the analysis of \( K_D \), which, see Theorem [3.5] also directly governs the convergence of a simplified two-level scheme with a single step of (damped) Jacobi smoothing.

The AMG algorithm first partitions the unknowns set into coarse and fine unknowns.
Next, a prolongation $P$ is set up that has the form

$$P = \begin{pmatrix} J_{FC} \\ I_{CC} \end{pmatrix},$$

where $I_{CC}$ is the identity for the coarse unknowns and $J_{FC}$ an interpolation operator for fine unknowns. For the considered class of matrices, the row sum of $J_{FC}$ should be 1 when the row sum of the corresponding row is zero; that is, everywhere in the singular case. Then, the range of $P$ contains the constant vector; i.e., the null space of $A$, and we can therefore apply our theoretical results with $\hat{P} = P$. In fact, in this case the “direct” interpolation rule [32, p 448] amounts to associated to each fine unknown $i$ a set $P_i$ of coarse neighboring unknowns, and set, for all $j$,

$$(J_{FC})_{ij} = \begin{cases} \frac{|a_{ij}|}{\sum_{k \in P_i} |a_{ik}|} & \text{if } j \in P_i \\ 0 & \text{otherwise} \end{cases}.$$

Now, the coarsening algorithms, that define the fine/coarse partitioning and select the set $P_i$ for each fine unknown, are actually designed in such a way that the following constant is only moderately larger than 1:

$$\tau = \max_{i \in F} \frac{\sum_{k \neq i} |a_{ik}|}{\sum_{k \in P_i} |a_{ik}|}.$$

This is motivated by theoretical considerations: one finds, for all $v \in \mathbb{R}^n$ (using Schwarz inequality and remembering that $a_{ij} \leq 0$ for $i \neq j$ whereas, in the singular case, $a_{ii} = \sum_{k \neq i} |a_{ik}|$ for all $i$),

$$\sum_{i \in F} a_{ii} \left( v_i - \sum_{j \in P_i} (J_{FC})_{ij} \mathbf{w}_j \right)^2 = \sum_{i \in F} a_{ii} \left( \sum_{j \in P_i} \frac{|a_{ij}|}{\sum_{k \in P_i} |a_{ik}|} (v_i - v_j) \right)^2 \leq \sum_{i \in F} a_{ii} \left( \sum_{j \in P_i} \frac{|a_{ij}|}{\sum_{k \in P_i} |a_{ik}|} \right) \left( \sum_{j \in P_i} \frac{|a_{ij}|}{\sum_{k \in P_i} |a_{ik}|} (v_i - v_j)^2 \right) \leq \tau \sum_{i \in F} \sum_{j \in P_i} |a_{ij}| (v_i - v_j)^2 \leq \tau \sum_{i \in F} \sum_{j \in C} |a_{ij}| (v_i - v_j)^2 \leq \frac{-\tau}{2} \sum_{i} \sum_{j} a_{ij} (v_i^2 - v_j^2) = \frac{-\tau}{2} \sum_{i} \sum_{j} a_{ij} (v_i^2 + v_j^2 - 2 \mathbf{w}_i \mathbf{w}_j) = \tau \mathbf{v}^T A \mathbf{v}.$$
And, with the help of Theorem 3.6, nothing more is needed to show that
\[ K_D \leq \tau. \]
Indeed, a valid “simplified” projector in (3.40) is
\[ \pi = I - P(0 \ A_{CC}) = \begin{pmatrix} I_{FF} & -J_{FC} \\ 0 & 0 \end{pmatrix}, \]
for which one has
\[ K_D(\pi) = \sup_{v \in \mathbb{R}^n, v \not\in N(A)} \frac{\sum_{i \in F} a_{ii} (v_i - \sum_{j \in P_i} (J_{FC})_{ij} w_j)^2}{v^T A v}. \]

4 Conclusions

We have provided a generalization to singular matrices of all the main theoretical results constitutive of the approach developed in \[1, 6, 13, 14, 27, 30, 32\]. As illustrated by the above discussion of classical AMG, the main outcome is that the transition from regular problems to singular ones runs smoothly. There are in fact many methods inspired by these results (e.g., \[6, 9, 26, 28, 30\]), and it is not possible to discuss here in detail their extension to singular problems. However, the results obtained in this paper suggest that, at least regarding the theoretical foundations, this extension should raise no particular difficulties. From that viewpoint the present theoretical study also complements the works where multilevel algorithms have been successfully applied to solve singular problems (e.g., \[3, 10, 11, 12, 21, 35, 36, 39\]).

On the other hand, regarding the coarse grid correction, we provided a detailed account on what’s going on when the coarse grid matrix is singular. We showed that “natural” singularities are harmless, defining these as those inherited from right singular modes in the range of the prolongation, or left singular modes in the range of the transpose of the restriction. Moreover, we showed that, in many cases, the course of the iterations is independent of the choice of the generalized inverse; i.e., independent of which particular solution of the coarse grid system is picked up. A noteworthy exception is, as arises in applications to Markov chains, when the range of the transpose of the restriction contains the left null space while the range of the prolongation does not contain the right null space. Then, the convergence may depend upon the choice of the generalized inverse, and some of our results (in particular Lemma 3.2) offer tools that can contribute to a detailed analysis of induced effects. Such an analysis, however, remains to be done.

References


[34] E. Treister, *private communication*.


